Portfolio Optimization with Random Liability in the Stochastic Interest Rate Environments

HAO CHANG
Tianjin Polytechnic University,
School of Science,
Tianjin 300387, CHINA;
Tianjin University,
College of Management and Economics,
Tianjin 300072, CHINA.
Email: ch8683897@126.com

WEI WEI
Tianjin Polytechnic University,
School of Science,
Tianjin 300387, CHINA;
Email: 593979032@qq.com

Abstract: This paper applies dynamic programming principle and Legendre transform to study a dynamic asset allocation problem with liability process and stochastic interest rate model, where interest rate is assumed to be driven by the Ho-Lee model or the Vasicek model. By using variable change technique, we obtain the closed-form solutions to the optimal investment strategies in the quadratic utility framework. Finally, a numerical example is given to analyze the impact of market parameters on the optimal investment strategies and some economic implications are provided. The numerical results imply that the amount invested in the stocks in the liability setting is larger than that in the no-liability setting.

Key–Words: the Ho-Lee model; the Vasicek model; asset and liability management(ALM); Legendre transform; optimal investment strategy;

1 Introduction

In the actual investment environment, it is found that interest rate is generally not fixed but dynamically changing, which can be delineated by some term structure models. The Ho-lee model [1] and the Vasicek model [2], which were used to describe the short term structure of interest rate, are the most important mathematical models. Recently, many scholars have investigated dynamic portfolio problems with stochastic interest rate, and obtained some instructive results. For example, using the stochastic optimal control theory, Korn and Kraft [3] studied the dynamic portfolio problem under the Ho-Lee model and the Vasicek model, and got the close-form expressions of the optimal investment strategies under power utility function. In addition, the verification theorem which verifies that a solution of the HJB equation is actually the optimal solution to the original optimization problem is presented, and laid a theoretical foundation for dynamic portfolio theory with stochastic interest rate. Fleming and Pang [4] focused on an investment and consumption problem under the Vasicek model, and proved the existence of the solution to the HJB equation by employing the sup-subsolution method, but the explicit expression of the optimal investment and consumption strategy is not given. Gao [5] investigated the pension fund problem in an affine interest rate framework and used a Legendre transform to obtain the explicit solutions to optimal investment strategies with logarithmic preference. Further, Noh and Kim [6] discussed the investment and consumption problem with stochastic interest rate and the stochastic volatility, but they didn't get the explicit solution of the optimal investment and consumption strategy. For more detailed models, some interested reader can refer to the works of Chang and Rong [7], Guan and Liang [8-9], Chang et al. [10], Zhang et al. [11], Liu et al. [12]. These models have greatly expanded the portfolio theory with stochastic interest rate. But they don't consider the factor that financial institutions or investors may be in debt in the process of investment. As a matter of fact, introducing the liability into the portfolio selection problems and investigating the optimal portfolio in the stochastic interest rate environments will be more practice.

It is well known that lots of financial institutions were often confronted with some liabilities in the process of investment, such as banks, insurance companies. The existence of the liability has certain effect on the investment strategies of financial institutions. In recent years, some scholars have studied the portfolio problems with random liability, and obtained some valuable results. Sharpe and Tint [13] established the mean-variance model with liability, and analyzed the optimal investment strategy and the relationship between return and risk. Browne [14] first studied the investment problem of insurance fund, and the optimal investment strategies for the insurers under maximizing exponential utility and minimizing the probability of bankruptcy were obtained. Leippold et al. [15] considered an asset-liability management problem in a muilt-period environment, and analyzed the impacts of liability on the optimal investment strategy and the characteristics of effective frontiers. Chiu and Li [16] got the optimal investment strategy in the continuous-time mean-variance framework and the explicit expression of the efficient frontier by applying stochastic linear-quadratic control method, where the liability process is driven by classical geometric Brownian motion. Xie et al. [17] assumed that the liability process is modeled by Brownian motion with a drift, and obtained the explicit solutions to the optimal investment strategy and the effective frontier, and analyzed the impacts of liability. Chen et al. [18] considered the management problem of asset-liability with regime-switching, and obtained the explicit expressions of the optimal investment strategy and the effective frontier. These models investigated the asset-liability management problems under different investment environments, and provided the theoretical basis for asset hedging and risk controlling in the liability settings. But these results have a drawback that these research results are obtained under the assumption of constant interest rate or particular investment constraints. It is more clear that investigating ALM problems with stochastic interest rate is of important academic value and broad prospect of application.

Mean-variance criterion is one of most important criteria in the portfolio selection theory. However, limitations of research methods greatly limited research progress of mean-variance models for a long time. Until 2000, Li and Ng [19] first presented a bedding technique and successfully solved a multi-period mean-variance model. This attracted some attentions of researchers and greatly promoted research progress on mean-variance models. Later, some scholars enriched and extended this technique and put forward some new methods to solve the mean-variance models. For example, Zhou and Li [20] presented a stochastic linear-quadratic(LQ) control method to deal with a continuous-time mean-variance model and obtained the closed-form solutions to the efficient strategy and the effective frontier. Li et al. [21] and Fu et al. [22] used the LQ technique and Lagrange duality theorem together to tackle the mean-variance model with constrains. Ferland and watier [23] and Shen et al. [24] used backward stochastic differential equation (BSDE) theory to solve the mean-variance model with stochastic market coefficients. But, if introducing liability process into above mentioned mean-variance models, above methods have some limitations and difficulties. Considering that the optimal portfolios under quadratic utility is mean-variance effective and may pave the way for solving mean-variance model, in this paper we devote ourselves to solving an ALM problem with stochastic interest rate in the quadratic utility framework.

Based on the works of Korn and Kraft [3], Xie et al.[17] and Zhou and Li [20], we introduce liability process into a continuous-time portfolio selection problem with stochastic interest rate and extend the relationship between liability dynamics and stock price dynamics to a general correlation coefficient. The financial market consists of one risk-free asset and multiple risky assets, where short rate is driven by the Ho-Lee model or the Vasicek model, and interest rate dynamics is generally linearly correlated with stock price dynamics. We study the optimal investment strategy under quadratic utility and obtain the explicit expressions of optimal investment strategies by applying dynamic programming principle and Legendre transform. Finally, a numerical example is provided to analyze the impact of market parameters on the optimal investment strategies, especially the liability parameters and interest rate parameters.

The rest of this paper is organized as follows. Section 2 presents the problem formulation of this paper. Section 3 uses the principle of stochastic dynamic programming to derive the HJB equation for the value function and applies Legendre transform to transform the original HJB equation into its dual equation. The explicit expressions of the optimal policies are obtained in Section 4. Section 5 gives a numerical illustration and Section 6 concludes the paper.

2 The model

The financial market, wealth process, liability process and optimization criterion are given in this section. Notation assumptions are as follows.

$(\cdot)'$ represents the transpose of a matrix or a vector, and $[0,T]$ stands for finite fixed investment horizon, and $E(\cdot)$ is the mathematical expectation. $\|x\| = \sqrt{x'x} = \sqrt{x_1^2+x_2^2+\cdots+x_n^2}$ represents the norm of the vector $x = (x_1, x_2, \cdots, x_n)'$, and $(W_1(t), W_2(t), \cdots, W_n(t))'$ is the n-dimension independent and standard Brownian motion defined on the complete probability space $(\Omega, \mathscr{F}, \mathbb{P}, \{\mathscr{F}_t\}_{0\leqslant t\leqslant T})$, where $\{\mathscr{F}_t\}_{0\leqslant t\leqslant T}$ can be interpreted as the information available at time t. Supposed that the financial market is composed of one risk-free asset and multiple risky assets.

One risk-free asset is interpreted as a bank account, whose price at time t is denoted by $P_0(t)$, Then $P_0(t)$ evolves according to the following equation

$$dP_0(t) = r(t)P_0(t)dt, \quad P_0(0) = 1, \tag{1}$$

where $r(t)$ is interest rate.

In this paper, we assume that $r(t)$ is a stochastic process satisfying the following term structure of interest rate

$$dr(t) = a(t)dt + bdW_r(t), \quad r(0) = r_0 > 0, \tag{2}$$

where $a(t)$ is the bounded function of time t, and $b > 0$ is a constant. $W_r(t)$ is a one-dimension standard Brownian motion defined on $(\Omega, \mathscr{F}, \mathbb{P}, \{\mathscr{F}_t\}_{0\leqslant t\leqslant T})$. If $a(t)$ is the function only about time t and is irrelevant to the interest rate $r(t)$, the equation (2) is called the Ho-Lee model. Assumed that $a(t)$ can be written as $a(t) = k(\alpha - r(t))$, where k and α are constants, the equation (2) is known as the Vasicek model.

Multiple risky assets are taken as the stocks, whose price of the *ith* stock at time t is denoted by $P_i(t)$, $i = 1, 2, \cdots, n$. Then the dynamics behavior of price process $P_i(t)$ can be described by the following geometric Brownian motion:

$$\begin{aligned} &dP_i(t) = P_i(t)\left((\mu_i + r(t))dt + \sum_{j=1}^{n} \sigma_{ij} dW_j(t)\right), \\ &P_i(0) = p_i > 0, \end{aligned} \tag{3}$$

where $\mu = (\mu_1, \mu_2, \cdots, \mu_n)'$ represents the appreciate rate vector of the stock. In addition, $\sigma = (\sigma_{ij})_{n\times n}$ represents the volatility matrix of the stock. Suppose that μ and σ are bounded constants, which are $\mathscr{F}_t-$measurable on the time horizon $[0,T]$, and satisfy the non-degeneracy condition: $\sigma\sigma' > 0$, $\forall t \in [0,T]$.

Suppose that an investor is equipped with an initial endowment $w_0 > 0$ and an initial liability $l_0 > 0$ at time $t = 0$, then the net initial wealth of an investor is $x_0 = w_0 - l_0$. Suppose that the accumulative liability at time t is denoted by $L(t)$, then $L(t)$ can be described by the following Brownian motion with drift:

$$dL(t) = udt + vdW_L(t), \quad L(0) = l_0 > 0, \tag{4}$$

where $u > 0$ and $v > 0$ are constants, and $W_L(t)$ is a one-dimension standard Brownian motion defined on $(\Omega, \mathscr{F}, \mathbb{P}, \{\mathscr{F}_t\}_{0\leqslant t\leqslant T})$.

In the process of the actual investment, the volatility of interest rate has some effect on the price of the stock, and the liability behavior of financial institutions also has some impact on the price of the stock. In this paper, we assume that interest rate process and the liability process are all generally correlated with the price of the stock. Suppose that the correlation coefficient between the volatility source of interest rate $W_r(t)$ and the volatility source of stock price $W_i(t)$ is denoted by ρ_i, and the correlation coefficient between the volatility source of the liability $W_L(t)$ and $W_i(t)$ is denoted by λ_i, then $W_r(t)$ and $W_L(t)$ can be expressed as:

$$W_r(t) = \sum_{i=1}^{n} \rho_i W_i(t) + \sqrt{1 - \|\rho\|^2}\tilde{W}_r(t),$$

$$W_L(t) = \sum_{i=1}^{n} \lambda_i W_i(t) + \sqrt{1 - \|\lambda\|^2}\tilde{W}_L(t),$$

where, $\tilde{W}_r(t)$ and $\tilde{W}_L(t)$ are two one-dimension independent and standard Brownian motions defined on $(\Omega, \mathscr{F}, \mathbb{P}, \{\mathscr{F}_t\}_{0\leqslant t\leqslant T})$.

Letting $\rho = (\rho_1, \rho_2, \cdots, \rho_n)'$, $\lambda = (\lambda_1, \cdots, \lambda_n)'$, $W(t) = (W_1(t), W_2(t), \cdots, W_n(t))'$, and $W(t)$ is independent of $\tilde{W}_r(t)$ and $\tilde{W}_L(t)$, then the interest rate process (2) and the liability process (4) can be written as

$$\begin{aligned} &dr(t) = a(t)dt + b\rho' dW(t) + b\sqrt{1 - \|\rho\|^2}d\tilde{W}_r(t), \\ &r(0) = r_0 > 0; \end{aligned} \tag{5}$$

$$\begin{aligned} &dL(t) = udt + v\lambda' dW(t) + v\sqrt{1 - \|\lambda\|^2}d\tilde{W}_L(t), \\ &L(0) = l_0 > 0. \end{aligned} \tag{6}$$

Assume that an investor has the net initial wealth $x_0 > 0$ at initial time $t = 0$, and the amount of invested in the *ith* stock at time t is denoted by $\pi_i(t)$, $i = 1, 2, \cdots, n$, then the amount invested in the risk-free asset is given by $\pi_0(t) = X(t) - \sum_{i=1}^{n} \pi_i(t)$, where $X(t)$ represents the net wealth of an investor at time

t. Letting $\pi(t) = (\pi_1(t), \pi_2(t), \cdots, \pi_n(t))'$, then the net wealth process satisfies

$$dX(t) =(X(t) - \sum_{i=1}^{n} \pi_i(t))\frac{dP_0(t)}{P_0(t)} + \sum_{i=1}^{n} \pi_i(t)\frac{dP_i(t)}{P_i(t)} - dL(t).$$

Taking the equation (1), (3) and (6) into consideration, we can get

$$dX(t) =\Big(r(t)X(t) + \pi'(t)\bar{\mu} - u\Big)dt + \Big(\pi'(t)\sigma - v\lambda'\Big)dW(t) - v\sqrt{1-\|\lambda\|^2}d\tilde{W}_L(t), \tag{7}$$

with the initial value $X(0) = x_0 > 0$, $\bar{\mu} = (\mu_1, \mu_2, \cdots, \mu_n)'$.

Definition 1 *(Admissible strategy). An investment strategy $\pi(t)$ is admissible if $\pi(t)$ satisfies the following conditions:*

(i) $\pi(t)$ is progressively $\mathscr{F}_t-$ measurable;

(ii) $E\Bigg(\displaystyle\int_t^T \Bigg(\|\pi'(t)\sigma - v\lambda'\|^2 + \Big(v\sqrt{1-\|\lambda\|^2}\Big)^2\Bigg)dt\Bigg) < \infty;$

(iii) For any investment strategy $\pi(t)$, the SDE (7) has a pathwise unique solution.

We denote the set of all admissible strategies $\pi(t)$ by $\prod = \{\pi(t) : 0 \leqslant t \leqslant T\}$, and the investor expects to find an optimal investment strategy to maximize the expected utility of terminal net wealth. Mathematically, the problem can be described as the following optimization problem:

$$\underset{\pi(t)\in\Pi}{Maximize}\ E[U(X(T))], \tag{8}$$

where $U(x)$ represents utility function and satisfies the conditions: the first-order derivative $\dot{U}(x) > 0$ and the second-order derivative $\ddot{U}(x) < 0$.

Due to the optimal investment strategy under the quadratic utility function is mean-variance effective. Thus, in this paper we mainly study the optimal investment strategy for the problem (8) with quadratic utility function. In utility theory of portfolio selection, the quadratic utility expression can be written as:

$$U(x) = x - \eta x^2, \quad x < 1/(2\eta),\ \eta > 0,$$

where η is the risk aversion factor.

3 HJB equation and Legendre transform

In this section, we study the problem (8) by applying dynamic programming principle and Legendre transform. Firstly, the HJB equation that the value function satisfies is derived by employing dynamic programming principle. Then, the dual equation to the value function is obtained by Legendre transform. Finally, we solve the dual equation by using variable change technique and get the explicit expression of the optimal investment strategy under quadratic utility function.

The problem (8) is considered to be a class of stochastic optimal control problems, then the value function $H(t,r,x)$ can be defined as:

$$H(t,r,x) = \sup_{\pi(t)\in\Pi} E[U(X(T))\,|X(t) = x, r(t) = r],$$

with boundary condition $H(T,r,x) = U(x)$.

According to the principle of stochastic dynamic programming, the value function $H(t,r,x)$ can be regarded as a continuous solution of the following HJB equation:

$$\begin{aligned}\sup_{\pi(t)\in\Pi} \Bigg[&H_t + (rx + \pi'(t)\bar{\mu} - u)H_x \\ &+ \frac{1}{2}\left(\left\|\pi'(t)\sigma - v\lambda'\right\|^2 + \Big(v\sqrt{1-\|\lambda\|^2}\Big)^2\right)H_{xx} \\ &+ a(t)H_r + \frac{1}{2}\Big(b^2\|\rho\|^2 + b^2(1-\|\rho\|^2)\Big)H_{rr} \\ &+ b\rho(\pi'(t)\sigma - v\lambda')H_{rx}\Bigg] = 0,\end{aligned} \tag{9}$$

where $H_t, H_r, H_{rr}, H_x, H_{xx}, H_{xr}$ represent the first-order and second-order partial derivatives of the value function $H(t,r,x)$ with respect to the variables t, r, x respectively.

According to the necessary condition of arriving at the maximum, we obtain the optimal value as follows

$$\pi^*(t) = -(\sigma\sigma')^{-1}\left(\bar{\mu}\frac{H_x}{H_{xx}} + b\sigma\rho\frac{H_{rx}}{H_{xx}}\right) + (\sigma')^{-1}\lambda v. \tag{10}$$

Letting $\theta = \sigma^{-1}\bar{\mu}$, and putting (10) into (9), we derive

$$\begin{aligned}&H_t + (rx - u + \theta' v\lambda)H_x + \frac{1}{2}v^2(1-\|\lambda\|^2)H_{xx} \\ &+ a(t)H_r + \frac{1}{2}b^2H_{rr} - \frac{1}{2}\|\theta\|^2\frac{H_x^2}{H_{xx}} \\ &- \frac{1}{2}b^2\|\rho\|^2\frac{H_{rx}^2}{H_{xx}} - b\theta'\rho\frac{H_xH_{rx}}{H_{xx}} = 0.\end{aligned} \tag{11}$$

It is very difficult for us to conjecture the structure of a solution to the equation (11) under quadratic utility function. So we introduce the following Legendre transform to derive its dual equation to (11).

Legendre transform can be defined by (referring to Gao [5]):

$$\hat{H}(t,r,z) = \sup\{H(t,r,x) - zx\},$$
$$g(t,r,z) = \inf\{x \mid H(t,r,x) \geqslant zx + \hat{H}(t,r,z)\}, \tag{12}$$

where $z > 0$ denotes the dual variable to x. The function $g(t,r,z)$ and $\hat{H}(t,r,z)$ can be seen as the dual function to $H(t,r,x)$. This paper chooses $g(t,r,z)$ as the dual function to $H(t,r,x)$.

The relationship between the function $\hat{H}(t,r,z)$ and $g(t,r,z)$ is as follows:

$$g(t,r,z) = -\hat{H}_z(t,r,z). \tag{13}$$

Notice that $H(T,r,x) = U(x)$, then at terminal time T, we define

$$\hat{H}(T,r,z) = \sup\{U(x) - zx\},$$
$$g(T,r,z) = \inf\{x \mid U(x) \geqslant zx + \hat{H}(T,r,z)\}. \tag{14}$$

So we have

$$g(T,r,z) = (\dot{U})^{-1}(z), \tag{15}$$

where $(\dot{U})^{-1}(z)$ is taken as the inverse of marginal utility.

According to the equation (12), we get

$$H_x(t,r,x) = z,$$

and we have:

$$g(t,r,z) = x, \quad \hat{H}(t,r,z) = H(t,r,g) - zg,$$

Referring to the work of Gao[5], we have

$$\begin{aligned} &H_t = \hat{H}_t, \quad H_x = z, \\ &H_{xx} = -\frac{1}{\hat{H}_{zz}}, \quad H_r = \hat{H}_r, \\ &H_{rr} = \hat{H}_{rr} - \frac{\hat{H}_{rz}^2}{\hat{H}_{zz}}, \quad H_{xr} = -\frac{\hat{H}_{rz}}{\hat{H}_{zz}}. \end{aligned} \tag{16}$$

Substituting (16) back into (11), then (11) can be written as:

$$\begin{aligned} &\hat{H}_t + rzg + (\theta' v\lambda - u)z - \frac{1}{2}v^2(1 - \|\lambda\|^2)\frac{1}{\hat{H}_{zz}} \\ &+ a(t)\hat{H}_r + \frac{1}{2}b^2\hat{H}_{rr} - \frac{1}{2}b^2(1 - \|\rho\|^2)\frac{\hat{H}_{rz}^2}{\hat{H}_{zz}} \\ &+ \frac{1}{2}\|\theta\|^2 z^2\hat{H}_{zz} - b\theta'\rho z\hat{H}_{rz} = 0. \end{aligned} \tag{17}$$

Differentiating (17) with respect to z and using (13), we obtain the dual equation

$$\begin{aligned} &g_t - rg + (\|\theta\|^2 - r)zg_z + u - \theta' v\lambda \\ &+ \frac{1}{2}v^2(1 - \|\lambda\|^2)\frac{g_{zz}}{g_z^2} + (a(t) - b\theta'\rho)g_r \\ &+ \frac{1}{2}b^2(1 - \|\rho\|^2)\frac{2g_r g_{rz} g_z - g_r^2 g_{zz}}{g_z^2} \\ &+ \frac{1}{2}\|\theta\|^2 z^2 g_{zz} - b\theta'\rho z g_{rz} = 0. \end{aligned} \tag{18}$$

Although (18) is a nonlinear partial differential equation, the structure of a solution to (18) is easy to conjecture under quadratic utility function. In the following section, we try our best to solve the equation (18) and obtain the optimal investment strategy for the original problem (8).

4 The optimal portfolios

Under quadratic utility function, the boundary condition of (18) should be

$$g(T,r,z) = (\dot{U})^{-1}(z) = \frac{1}{2\eta}(1 - z).$$

Thus, assume that a solution to (18) is conjectured as follows:

$$\begin{aligned} &g(t,r,z) = f(t,r)z + h(t,r), \\ &f(T,r) = -\frac{1}{2\eta}, \quad h(T,r) = \frac{1}{2\eta}. \end{aligned} \tag{19}$$

Further, the partial derivatives of $g(t,r,z)$ with respect to t, r, and z are as follows:

$$\begin{aligned} &g_t = f_t z + h_t, \quad g_r = f_r z + h_r, \\ &g_z = f, \quad g_{rr} = f_{rr}z + h_{rr}, \\ &g_{zz} = 0, \quad g_{rz} = f_r. \end{aligned} \tag{20}$$

Plugging (20) into (18), after some simple calculations, we derive

$$z\left[f_t+(\|\theta\|^2-2r)f+(a(t)-2b\theta'\rho)f_r+\frac{1}{2}b^2f_{rr}-b^2(1-\|\rho\|^2)\frac{f_r^2}{f}\right]+h_t-rh+u-\theta'v\lambda+(a(t)-b\theta'\rho)h_r+\frac{1}{2}b^2h_{rr}-b^2(1-\|\rho\|^2)\frac{f_r}{f}h_r=0.$$

We can decompose the above equation into the following two equations in order to eliminate the dependence on z:

$$f_t+(\|\theta\|^2-2r)f+(a(t)-2b\theta'\rho)f_r+\frac{1}{2}b^2f_{rr}-b^2(1-\|\rho\|^2)\frac{f_r^2}{f}=0; \tag{21}$$

$$h_t-rh+u-\theta'v\lambda+(a(t)-b\theta'\rho)h_r+\frac{1}{2}b^2h_{rr}-b^2(1-\|\rho\|^2)\frac{f_r}{f}h_r=0. \tag{22}$$

Lemma 2 *Assume that a solution to (21) is conjectured as $f(t,r)=A(t)e^{B(t)r}$, with boundary conditions given by $A(T)=-\frac{1}{2\eta}$ and $B(T)=0$, then we obtain the following results:*

(i) Under the Ho-Lee model: $B(t)$ and $A(t)$ are determined by (24) and (25), respectively.

(ii) Under the Vasicek model: $B(t)$ and $A(t)$ are given by (26) and (27), respectively.

Proof. Substituting $f(t,r)=A(t)e^{B(t)r}$ into (18), then the equation (18) can be written as

$$A(t)e^{B(t)r}\left[\frac{\dot{A}(t)}{A(t)}+\|\theta\|^2+(a(t)-2b\theta'\rho)B(t)+\frac{1}{2}b^2(2\|\rho\|^2-1)B^2(t)+r(\dot{B}(t)-2)\right]=0. \tag{23}$$

(i) Under the Ho-Lee model

In the Ho-Lee model, $a(t)$ is a bounded deterministic function about time t and is irrelevant to interest rate $r(t)$. Thus, the equation (23) can be decomposed into the following two ordinary differential equations:

$$\dot{B}(t)-2=0,\quad B(T)=0;$$

$$\frac{\dot{A}(t)}{A(t)}+\|\theta\|^2+(a(t)-2b\theta'\rho)B(t)+\frac{1}{2}b^2(2\|\rho\|^2-1)B^2(t)=0,\quad A(T)=-\frac{1}{2\eta}.$$

Solving the above equation, we obtain

$$B(t)=2(t-T), \tag{24}$$

$$A(t)=-\frac{1}{2\eta}\exp\left\{\int_t^T\left(\|\theta\|^2+(a(t)-2b\theta'\rho)B(t)+\frac{1}{2}b^2(2\|\rho\|^2-1)B^2(t)\right)dt\right\}. \tag{25}$$

(ii) Under the Vasicek model

In the Vasicek model, $a(t)$ can be written as $a(t)=k(\alpha-r)$. Comparing the coefficients on both sides of the equation (23), we have

$$\dot{B}(t)-kB(t)-2=0,\quad B(T)=0;$$

$$\frac{\dot{A}(t)}{A(t)}+\|\theta\|^2+(k\alpha-2b\theta'\rho)B(t)+\frac{1}{2}b^2(2\|\rho\|^2-1)B^2(t)=0,\quad A(T)=-\frac{1}{2\eta}.$$

Solving the above two equations, we get

$$B(t)=\frac{2}{k}(e^{-k(T-t)}-1), \tag{26}$$

$$A(t)=-\frac{1}{2\eta}\exp\left\{\int_t^T\left(\|\theta\|^2+(k\alpha-2b\theta'\rho)B(t)+\frac{1}{2}b^2(2\|\rho\|^2-1)B^2(t)\right)dt\right\}. \tag{27}$$

As a result, Lemma 2 is completed. ■

Notice that $u-\theta'v\lambda$ is a constant, and is irrelevant to the variable t. Thus, we have the following lemma.

Lemma 3 *Suppose that*

$$h(t,r)=(u-\theta'v\lambda)\int_t^T\hat{h}(s,r)ds+\frac{1}{2\eta}\hat{h}(t,r),$$

then the equation (22) can be changed into:

$$\hat{h}_t-r\hat{h}+(a(t)-b\theta'\rho)\hat{h}_r+\frac{1}{2}b^2\hat{h}_{rr}-b^2(1-\|\rho\|^2)\frac{f_r}{f}\hat{h}_r=0,\quad \hat{h}(T,r)=1. \tag{28}$$

Proof. Introducing the following variational operator on any function $h(t,r)$:

$$\nabla h(t,r)=-rh+(a(t)-b\theta'\rho)h_r+\frac{1}{2}b^2h_{rr}-b^2(1-\|\rho\|^2)\frac{f_r}{f}h_r.$$

Then (22) can be rewritten as

$$\frac{\partial h(t,r)}{\partial t} + \nabla h(t,r) + u - \theta' v\lambda = 0. \quad (29)$$

Considering

$$h(t,r) = (u - \theta' v\lambda)\int_t^T \hat{h}(s,r)ds + \frac{1}{2\eta}\hat{h}(t,r),$$

then we have

$$\begin{aligned}\frac{\partial h(t,r)}{\partial t} =& -(u-\theta' v\lambda)\hat{h}(t,r) + \frac{1}{2\eta}\cdot\frac{\partial \hat{h}(t,r)}{\partial t}\\ =&(u-\theta' v\lambda)\left(\int_t^T \frac{\partial \hat{h}(s,r)}{\partial s}ds - \hat{h}(T,r)\right)\\ &+\frac{1}{2\eta}\cdot\frac{\partial \hat{h}(t,r)}{\partial t},\end{aligned}$$

$$\nabla h(t,r) = (u-\theta' v\lambda)\int_t^T \nabla\hat{h}(s,r)ds + \frac{1}{2\eta}\nabla\hat{h}(t,r).$$

So (29) can be converted into:

$$\begin{aligned}&(u-\theta' v\lambda)\Bigg(\int_t^T\left(\frac{\partial \hat{h}(s,r)}{\partial s} + \nabla\hat{h}(s,r)\right)ds\\ &+1-\hat{h}(T,r)\Bigg) + \frac{1}{2\eta}\left(\frac{\partial \hat{h}(t,r)}{\partial t} + \nabla\hat{h}(t,r)\right) = 0.\end{aligned}$$

Comparing the coefficients, we find that $\hat{h}(t,r)$ satisfies the following equation:

$$\frac{\partial \hat{h}(t,r)}{\partial t} + \nabla\hat{h}(t,r) = 0, \quad \hat{h}(T,r) = 1.$$

As a result, Lemma 3 is completed. ■

Lemma 4 *Letting $\hat{h}(t,r) = e^{D_1(t)+D_2(t)r}$ be the solution to the equation (28), where boundary conditions are given by $D_1(T) = 0$ and $D_2(T) = 0$, then we have the following conclusions:*

(i) Under the Ho-Lee model: $D_2(t)$ and $D_1(t)$ are determined by (31) and (32), respectively.

(ii) Under the Vasicek model: $D_2(t)$ and $D_1(t)$ are given by (33) and (34), respectively.

Proof. Putting $\hat{h}(t,r) = e^{D_1(t)+D_2(t)r}$ in the equation (28) yields

$$\begin{aligned}&e^{D_1(t)+D_2(t)r}\Bigg[\dot{D}_1(t) + \Bigg(a(t) - b\theta'\rho - b^2(1-\|\rho\|^2)\\ &*B(t)\Bigg)D_2(t) + \frac{1}{2}b^2D_2^2(t) + r(\dot{D}_2(t)-1)\Bigg] = 0. \quad (30)\end{aligned}$$

(i) Under the Ho-Lee model

Comparing the coefficients on both sides of the equation (30), we get the following two ordinary differential equations:

$$\dot{D}_2(t) - 1 = 0, \quad D_2(T) = 0;$$

$$\begin{aligned}&\dot{D}_1(t) + \Big(a(t) - b\theta'\rho - b^2(1-\|\rho\|^2)B(t)\Big)D_2(t)\\ &+\frac{1}{2}b^2D_2^2(t) = 0, \quad D_1(T) = 0.\end{aligned}$$

Solving the above equations, we derive

$$D_2(t) = (t-T), \quad (31)$$

$$\begin{aligned}D_1(t) =& \int_t^T\Bigg(\Big(a(t) - b\theta'\rho - b^2(1-\|\rho\|^2)B(t)\Big)\\ &\times D_2(t) + \frac{1}{2}b^2D_2^2(t)\Bigg)dt. \quad (32)\end{aligned}$$

where $B(t) = 2(t-T)$.

(ii) Under the Vasicek model

In the Vasicek model, $a(t)$ can be expressed as $a(t) = k(\alpha - r)$. So, the equation (30) can be decomposed into

$$\dot{D}_2(t) - kD_2(t) - 1 = 0, \quad D_2(T) = 0;$$

$$\begin{aligned}&\dot{D}_1(t) + \Big(k\alpha - b\theta'\rho - b^2(1-\|\rho\|^2)B(t)\Big)D_2(t)\\ &+\frac{1}{2}b^2D_2^2(t) = 0, \quad D_1(T) = 0.\end{aligned}$$

After some simple calculations, we have

$$D_2(t) = \frac{1}{k}(e^{-k(T-t)} - 1); \quad (33)$$

$$\begin{aligned}D_1(t) =& \int_t^T\Bigg(\Big(k\alpha - b\theta'\rho - b^2(1-\|\rho\|^2)B(t)\Big)\\ &\times D_2(t) + \frac{1}{2}b^2D_2^2(t)\Bigg)dt. \quad (34)\end{aligned}$$

where $B(t) = \frac{2}{k}(e^{-k(T-t)} - 1)$.

Therefore, we complete the proof of Lemma 4. ■

Further, according to the conclusions of (16), (19) and Lemma 2, we get

$$\begin{aligned}\frac{H_x}{H_{xx}} =& z(-\hat{H}_{zz}) = zg_z\\ =& zf = g - h = x - h,\\ \frac{H_{xr}}{H_{xx}} =& \hat{H}_{rz} = -g_r = -(f_r z + h_r)\\ =& -B(t)(x-h) - h_r.\end{aligned}$$

In short, the optimal investment strategies for the problem (8) under quadratic utility function can be summarized as the following conclusion.

Theorem 5 *Provided that quadratic utility function is expressed as $U(x) = x - \eta x^2$, where $x < \frac{1}{2\eta}$ and $\eta > 0$, then the optimal investment strategy is given by*

$$\begin{aligned}\pi^*(t) =& -(\sigma\sigma')^{-1}\Big(\bar{\mu}(X(t)-h) + (\sigma')^{-1}\lambda v \\ & - b\rho\sigma(B(t)(X(t)-h)+h_r)\Big),\end{aligned} \tag{35}$$

where,

$$\begin{aligned}h = h(t,r) =& (u-\theta' v\lambda)\int_t^T e^{D_1(s)+D_2(s)r(s)}ds \\ & + \frac{1}{2\eta}e^{D_1(t)+D_2(t)r(t)},\end{aligned}$$

$$\begin{aligned}h_r =& (u-\theta' v\lambda)\int_t^T D_2(s)e^{D_1(s)+D_2(s)r(s)}ds \\ & + \frac{1}{2\eta}D_2(t)e^{D_1(t)+D_2(t)r(t)}.\end{aligned}$$

And we have:

(i) Under the Ho-Lee model: $D_2(t)$ and $D_1(t)$ are determined by (31) and (32), respectively, $B(t) = 2(t-T)$.

(ii) Under the Vasicek model: $D_2(t)$ and $D_1(t)$ are determined by (33) and (34), respectively, $B(t) = 2(e^{-k(T-t)}-1)/k$.

Remark 6 *From (35), we find that the optimal investment strategy for the problem (8) with stochastic interest rate and random liability can be decomposed into three parts:*

$$\begin{aligned}& -(\sigma\sigma')^{-1}\bar{\mu}(X(t)-h), \\ & (\sigma\sigma')^{-1}b\sigma\rho(B(t)(X(t)-h)+h_r), \\ & (\sigma')^{-1}\lambda v,\end{aligned}$$

where, $-(\sigma\sigma')^{-1}\bar{\mu}(X(t)-h)$ is determined by the parameters of the stock, but is affected by the parameters of interest rate and the liability. And $(\sigma\sigma')^{-1}b\sigma\rho(B(t)(X(t)-h)+h_r)$ influenced by the stock and the liability factor is determined by the stochastic interest rate model. $(\sigma')^{-1}\lambda v$ is determined by the parameters of the liability model, on which the parameter of the stock has some impact.

In order to compare Theorem 5 with the conclusions of existing literatures, we analyze some special cases as follows.

Special case 1. If we don't consider liability factor, that is, $u = v = \lambda_i = 0$, $i = 1, 2, \ldots, n$, then the optimal policy for the problem (8) is given by

$$\begin{aligned}\pi^*(t) =& -(\sigma\sigma')^{-1}\Big(\bar{\mu}(X(t)-h) \\ & - b\sigma\rho(B(t)(X(t)-h)+h_r)\Big),\end{aligned} \tag{36}$$

where

$$\begin{aligned}h = h(t,r) &= \frac{1}{2\eta}e^{D_1(t)+D_2(t)r(t)}, \\ h_r &= \frac{1}{2\eta}D_2(t)e^{D_1(t)+D_2(t)r(t)}\end{aligned}$$

and we have:

(i) Under the Ho-Lee model: $D_2(t)$ and $D_1(t)$ are given by (31) and (32) respectively, and $B(t) = 2(t-T)$.

(ii) Under the Vasicek model: $D_2(t)$ and $D_1(t)$ are determined by (33) and (34) respectively, and $B(t) = 2(e^{-k(T-t)}-1)/k$.

Special case 2. If interest rate is a constant, that is to say, we have: $a(t) = 0$, $b = 0$, $k = 0$, $\rho_j = 0$, $j = 1, 2, \ldots, n$, then we get

$$D_1(t) = 0, \quad D_2(t) = t - T.$$

So the optimal strategy of the problem (8) is given by

$$\pi^*(t) = -(\sigma\sigma')^{-1}\bar{\mu}(X(t)-h) + (\sigma')^{-1}\lambda v, \tag{37}$$

where

$$h = h(t,r) = (u-\theta' v\lambda)\int_t^T e^{-r(T-t)}dt + \frac{1}{2\eta}e^{-r(T-t)}.$$

Special case 3. If there is no liability and interest rate is a constant , then the optimal investment policy of the problem (8) is given by

$$\pi^*(t) = -(\sigma\sigma')^{-1}\bar{\mu}\Big(X(t) - \frac{1}{2\eta}e^{-r(T-t)}\Big).$$

5 Numerical analysis

In order to analyze the effect of interest rate and liability on the optimal investment strategy, we provide a numerical example in this section. We assume that the financial market consists of one risk-free asset and two risky assets. Throughout this section, unless otherwise stated, the basic parameters are given by as follows: Under the Ho-Lee model, $a(t) = 0.035$, $b = 0.1$. Under the Vasicek model, $k = 0.5$, $\alpha = 0.07$, and $b = 0.1$. The initial value of interest rate $r(0) = 0.05$, $\rho = (0.7,\ 0.5)'$. In the liability model, $u = 0.1$, $v = 0.2$, and $\lambda(t) = (0.4,\ 0.6)'$. For the model of the stock price, $\mu = (0.13,\ 0.20)'$, $\sigma = \begin{pmatrix} 0.35 & 0.48 \\ 0.48 & 0.45 \end{pmatrix}$. Other parameters are as follows: $t = 0$, $T = 1$, $x_0 = 100$, and $\eta = 1/1000$. In the following pictures, we change one parameter and keep other parameters fixed, and analyze the impact of the main parameters on the optimal investment policy. The Y-axis in Figure 1-4 represents the sum of the amount invested in two stocks, which is denoted by $\sum_{i=1}^{2} \pi_i(t)$.

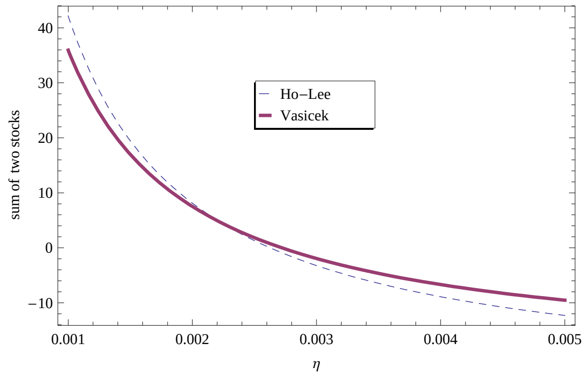

Figure 1: The impact of η on $\sum_{i=1}^{2} \pi_i(t)$.

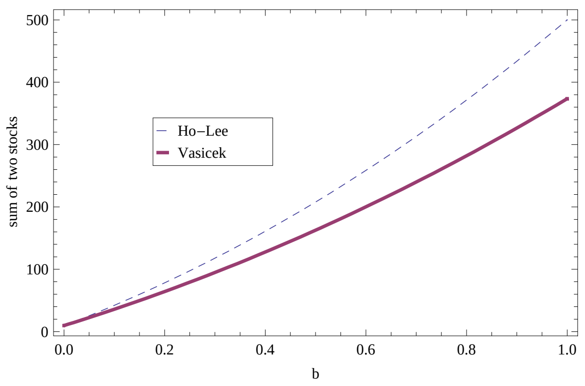

Figure 2: The impact of b on $\sum_{i=1}^{2} \pi_i(t)$.

From Fig.1-4, we can draw some instructive and valuable conclusions as follows.

(i) Both in the Ho-Lee model and in the Vasicek model, $\sum_{i=1}^{2} \pi_i(t)$ increases with respect to (w.r.t) the parameters b, u and v respectively. From the economic implications of b, u and v, this conclusion is obvious. In fact, the parameter b stands for the volatility of interest rate. It implies that a larger value of b means the more risk resulted from interest rate, which leads to the less amount invested in the risk-free asset. Correspondingly, the more money is invested in the two stocks. When the value of u is increasing, the expected value of liability is ascending. In order to asset hedge,

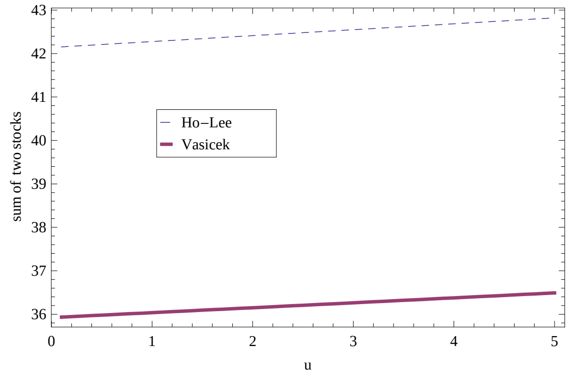

Figure 3: The impact of u on $\sum_{i=1}^{2} \pi_i(t)$.

the investor should hold more shares in the two stocks. The parameter v represents the volatility of liability. A greater value of v displays the more risks resulted from the liability. In order to hedge the risks from the liability, the investor should invest more money in the two stocks. On the other hand, the amount invested in risky assets in the liability setting is much larger than that in the no-liability setting.

(ii) Both in the Ho-Lee model and in the Vasicek model, $\sum_{i=1}^{2} \pi_i(t)$ decreases w.r.t the parameter η. Under quadratic utility, the risk aversion coefficient of an investor is denoted by $2\eta x/(1 - 2\eta x)$. When the value of η is increasing, the risk aversion agree of investors is increasing. It displays that an investor would like to investing more amount of wealth in the risk-free asset and investing less amount in the risky assets.

(iii) $\sum_{i=1}^{2} \pi_i(t)$ is very sensitive to the parameter b and η, while isn't very sensitive to the parameter u and v.

(iv) Based on the given numerical example, we come to the conclusion that when η is less than about 0.002, the amount invested in the stocks under the Ho-Lee model is larger than that under the Vasicek model. When $\eta > 0.002$, the conclusion is opposite to it.

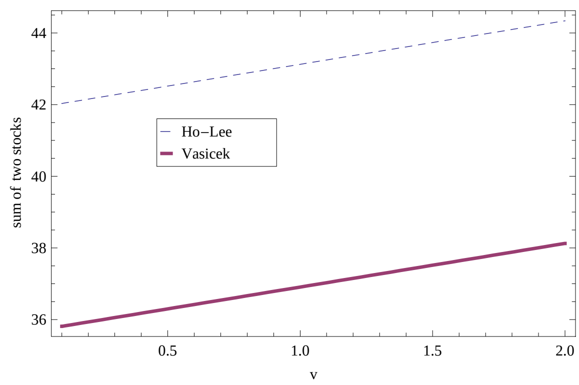

Figure 4: The impact of v on $\sum_{i=1}^{2} \pi_i(t)$.

6 Conclusions

In the practice environment of investments, interest rate should be a stochastic process which satisfies a certain term structure. In this paper, we assume that the dynamic behavior of interest rate can be described by the Ho-Lee model or the Vasicek model, and study the optimal investment problem with stochastic interest rate dynamics in the quadratic utility framework. The liability process is assumed to be driven by Brownian motion with drift and is generally correlated with stock price dynamics. By applying dynamic programming principle and Legendre transform, the closed-form solutions to the optimal investment strategies with quadratic utility preference are obtained. We also present a numerical example to analyze the influence of market parameters on the optimal investment strategies, which demonstrates that when $\eta < 0.002$, the amount invested in the stocks under the Ho-Lee model is larger than that under the Vasicek model, and the amount invested in risky assets in the liability setting is much larger than that in the no-liability setting. Theoretically, the results in this paper pave the way for solving the mean-variance portfolio selection problems with random liability and stochastic interest rate.

In further research, we can study some more sophisticated ALM problems in the mean-variance framework, for example, the ALM problem with interest rate and inflation risk, the ALM problem with stochastic interest rate and stochastic volatility, or the ALM problem with interest rate and Markovian regime-switching. It is very difficult for us to solve these problems at this stage. We leave these problems to future research.

Acknowledgements: This research is supported by China Postdoctoral Science Foundation Funded Project (No. 2014M560185), Tianjin Natural Science Foundation (No. 15JCQNJC04000).

References:

[1] T.S.Y. Ho and S.B. Lee, Term structure movements and pricing interest contingent claims, *Journal of Finance*, 41, 1986, pp. 1011–1029.

[2] O.A. Vasicek, An equilibrium characterization of the term structure, *Journal of Financial Economics*, 5, 1977, pp. 177–188.

[3] R. Korn and H. Kraft, A stochastic control approach to portfolio problems with stochastic interest rates, *SIAM Journal of Control and optimization*, 40, 2001, pp. 1250–1269.

[4] W.H. Fleming and T. Pang, An application of stochastic control theory to financial economics, *SIAM Journal on Control and Optimization*, 43, 2004, pp. 502–531.

[5] J.W. Gao, Stochastic optimal control of DC pension funds, *Insurance: Mathematics and Economics*, 42, 2008, pp. 1159–1164.

[6] E.J. Noh and J.H. Kim, An optimal portfolio model with stochastic volatility and stochastic interest rate, *Journal of Mathematical Analysis and Applications*, 375, 2011, pp. 510–522.

[7] H. Chang and X.M. Rong, An investment and consumption problem with CIR interest rate and stochastic volatility, *Abstract and Applied Analysis*, 2013, 2013, pp. 1–12, http://dx.doi.org/10.1155/2013/219397.

[8] G.H. Guan and Z.X. Liang, Optimal reinsurance and investment strategies for insurer under interest rate and inflation risks, *Insurance: Mathematics and Economics*, 55, 2014a, pp. 105–115.

[9] G.H. Guan and Z.X. Liang, Optimal management of DC pension plan in a stochastic interest rate and stochastic volatility framework, *Insurance: Mathematics and Economics*, 57, 2014b, pp. 58–66.

[10] H. Chang, X.M. Rong and H. Zhao, Optimal investment and consumption decisions under the Ho-Lee interest rate model, *WSEAS Transactions on Mathematics*, 12, 2013, pp. 1065–1075.

[11] L.D. Zhang, X.M. Rong and Z.P. Du, Optimal time-consistent investment in the dual risk model with diffusion, *WSEAS Transactions on Mathematics*, 14, 2015, pp. 213–225

[12] W.J. Liu, S.H. Fan and H. Chang, Dynamic optimal portfolios with CIR interest rate under a Heston model, *WSEAS Transactions on Systems and Control*, 10, 2015, pp. 421–429.

[13] W.F. Sharpe and L.G. Tint, Liabilities-a new approach, *Journal of Portfolio Management*, 16, 1990, pp. 5–10.

[14] S. Browne, Optimal investment policies for a firm with random risk process: exponential utility and minimizing the probability of ruin, *Mathematics of Operations Research*, 20, 1995, pp. 937-958.

[15] M. Leippold, F. Trojani and P. Vanini, A geometric approach to multi-period mean-variance optimization of assets and liabilities, *Journal of Economics Dynamics and Control*, 28, 2004, pp. 1079–1113.

[16] M.C. Chiu and D. Li, Asset and liability management under a continuous-time mean–variance optimization framework, *Insurance: Mathematics and Economics*, 39, 2006, pp. 330–355.

[17] S.X. Xie, Z.F. Li and S.Y. Wang, Continuous-time portfolio selection with liability: mean–variance model and stochastic LQ approach, *Insurance: Mathematics and Economics*, 42, 2008, pp. 943–953.

[18] P. Chen, H. Yang and G. Yin, Markowitz's mean-variance asset-liability management with regime switching: a continuous-time model, *Insurance: Mathematics and Economics*, 43, 2008, pp. 456–465.

[19] D. Li and W.L. Ng, Optimal dynamic portfolio selection: multi-period mean-variance formulation, *Mathematical Finance*, 10, 2000, pp. 387–406.

[20] X.Y. Zhou and D. Li, Continuous time mean-variance portfolio selection: A stochastic LQ framework, *Applied Mathematics and Optimization*, 42, 2000, pp. 19–33.

[21] X. Li, X.Y. Zhou and A.E.B. Lim, Dynamic mean-variance portfolio selection with no-shorting constraints, *SIAM Journal on Control and Optimization*, 40, 2002, pp. 1540–1555.

[22] C. Fu, A. Lari-Lavassani and X. Li, Dynamic mean-variance portfolio selection with borrowing constraint, *European Journal of Operational Research*, 200, 2010, pp. 312–319

[23] R. Ferland and F. Watier, Mean-variance efficiency with extended CIR interest rates, *Applied Stochastic Models in Business and Industry*, 26, 2010, pp. 71–84.

[24] Y. Shen, X. Zhang and T.K. Siu, Mean-variance portfolio selection under a constant elasticity of variance model, *Operations Research Letters*, 42, 2014, pp. 337–342.