Feedback Zero-Sum Linear Quadratic Dynamic Game for Descriptor System

SALMAH
Department Mathematics
Gadjah Mada University
Sekip Utara Yogyakarta Indonesia 55281
syalmah@yahoo.com http://matematika.fmipa.ugm.ac.id/salmah/

Abstract: - In this paper we present a Nash equilibrium problem of linear quadratic zero-sum dynamic games for descriptor system. We assume that the players give a linear feedback to the game. For the game with finite planning horizon we derive a differential Riccati type equation. For the game with infinite planning horizon we consider an algebraic Riccati type equation. The connection of the game solution with solution of Riccati equation will be studied. Furthermore for the game with infinite planning horizon we will derive numerical formulae for optimal Nash solution with invariant subspace method. The numerical formulae can be developed by considering generalized eigen vector that arised from descriptor system of the closed loop system.

Key-Words: - Nash equilibrium, feedback, zero-sum, linear quadratic, dynamic game, descriptor system

1 Introduction

In the last decades, there has been increasing research in economics with dynamic game approach. Especially, it is natural to model environmental economic and macro-economic policy coordination problems as dynamic game [2,10]. With this approach, the effect of the execution control strategy of the game to dynamic of the model can be analyzed [2, 4, 9, 10, 12]. In applications one often encounters systems described by differential equations system subject to algebraic constraints. The descriptor systems, gives a realistic model for this systems [3, 14, 16, 18, 19, 20]. The properties of descriptor system is studied in [5, 6, 7, 17].

In policy coordination problems, questions arise, are policies coordinated and which information do the parties have. One scenario is feedback Nash. According this, the parties can react to each other's policies, therefore it has large economic relevance [10].

In optimal control theory, it is well known to consider robust control strategies with dynamic game approach. In the study we need to consider a special game that is the zero-sum game [1]. If we want to design robust control system with dynamic game approach for descriptor system, we need to study the zero-sum game for descriptor system. Furthermore to consider robust control design that we can give feedback to the system, we need to consider the game with feedback scenario.

Such first step studies of differential game for descriptor systems have been carried out with assumption that the game is open loop. Hamiltonian method was used to study necessary and sufficient condition for existence of Nash solution of the game. To find the optimal solution of the dynamic game for descriptor system with a finite planning horizon, the problem is related to the solution of differential Riccati equation. For infinite horizon game the differential Riccati equation become algebraic Riccati equation [8, 24, 25, 26]. The Riccati-type equation is a generalization and combination of Riccati-type equation for linear quadratic dynamic game with standard system [10] and differential Riccati equation for linear quadratic optimal control for descriptor system [15, 21, 22]. The work of open-loop linear quadratic dynamic game for descriptor system with finite and infinite horizon case and Riccati-type equation for the game is in [24]. A simplifying assumption can be made, namely descriptor system with index one [8].

The purpose of this paper is to extend the investigation in [8, 24] where the game is non-zero-sum. In those studies, the assumption for the game is open-loop. Open-loop game is a benchmark to study more complicated game. This strategy is based on assumption that the parties act non-cooperatively and the only information they have is it present state and the model structure. In this scenario the parties can not react each other. Therefore its relevance is limited.

The zero-sum game with open-loop scenario was consider in [23]. The feedback game with descriptor system was consider in [11]. The paper consider the feedback descriptor game by applying transformation to the pencil $\lambda E - A$ and then reduce the problem into standard game problem. While in this paper we consider the feedback descriptor game directly without the pencil transformation. Because we consider the descriptor system directly without transform it into slow subsystem and fast subsystem, it can preserve structure of physical system.

In this paper we will study the game that including feedback Nash, in which the parties can react to each other's policies. This sceanario has large application relevance. The result from non-zero-sum game with linear feedback strategy will be applied to zero-sum game with linear feedback strategy.

In this paper we will consider a linear feedback zero-sum dynamic game in which the player satisfy a linear descriptor system and minimize quadratic objective function. For finite horizon problem, solution of generalized Riccati differential equation is studied. If the planning horizon is extended to infinity the differential Riccati equation will become an algebraic Riccati equation. Particular attention will be given to computational aspect of the problem. Without loss the generality, in this paper we consider game with two players.

2 Problem Formulation

Linear quadratic dynamic game can be considered as a combination of linear quadratic optimal control and game theory. In linear quadratic dynamic game, the parties (called players) try to minimize their individual quadratic objective function and give control to standard state space system.

Although it has many applications, ordinary linear quadratic optimal control, often does not provide a physical meaning in controlling, because the state variable does not corresponds with variable that we want to control. Descriptor systems have great capacity for system modeling and can include nondynamic mode and impulsive mode. Therefore they have a potential applicability for a wide class of systems. Descriptor system described by a set of ordinary equations subject to some algebraic constraints.

In linear quadratic dynamic game for descriptor system we consider the problem of players who like to optimize their quadratic cost function performance depending both on the state and control variables. The system is described by a set of differential and algebraic equations which is called a descriptor system. The game with two players that give control to descriptor system can be expressed mathematically as

$$E\dot{x} = Ax + B_1u_1 + B_2u_2,\ Ex(0) = Ex_0 , \quad (1)$$

with $E \in \Re^{nxn}, A \in \Re^{nxn}, B_1 \in \Re^{nxm_1}, B_2 \in \Re^{nxm_2}$, *x(t)* descriptor vector n dimension, while $u_i(t)$, $i=1,\dots,n$ are control vector m_i dimension which is done by i-th player, $i=1,\dots,n$. Matrix E generally singular with rank $E = r < n.$ The players minimizing in the Nash sense objective functions with the form

$$J_i(u_1,u_2) = \frac{1}{2}x(T)^T E^T K_{iT} Ex(T)$$
$$+\frac{1}{2}\int_0^T \left(x(t)^T Q_i x(t) + u_1^T(t)R_{i1}u_1(t) + u_2^T R_{i2}u_2(t)\right)dt,$$
$$i=1,2 \quad (2)$$

with all matrices symmetric. Furthermore Q_i and K_{if} semi positive definite and R_{ij} positive definite. In this paper we will consider a linear feedback strategy of the linear quadratic dynamic game for descriptor system. Below is definition of feedback strategy.

Definition 1: The set of control actions

$$F^* = (F_1^*, F_2^*)$$

is called a feedback Nash equilibrium if for all 1=1,2

$$J_i(x_0, F^*) \le (x_0, F_{-1}^*(\alpha)) \quad (3)$$

for every consistent x_0 and for each matrix α such that $F_{-1}^*(\alpha) \in F_N$.

Under some assumptions such as regularity, impulse controllability and index one we will solve the game, both for finite and in finite planning horizon. The assumptions that we use for descriptor system is given in Assumption 1.

Assumption 1: Descriptor system (1) regular, impulse controllable and finite dynamic stabilizable which satisfy

(i). $|sE - A| \ne 0, \forall s \ne 0$, except for a finite number of $s \in \Re$,

(ii). $\operatorname{Im} E + \operatorname{Im} A(\ker E) + \operatorname{Im}\left(B_1 | B_2\right) = \Re^n$,

(iii). $rank\left(sE - A \quad B_1 | B_2\right) = n \ \forall s, \operatorname{Re}[s] \ge 0$.

To find solution of linear quadratic dynamic game for descriptor system with finite horizon case, a differential Riccati equation will be derived. The relationship between the solution existence of differential Riccati equation and solution existence of the game will be considered. For infinite horizon case algebraic Riccati equation that associated with the game will be studied.

Now we will initiate the zero-sum game. Consider the problem that the players satisfy (1) and (2). The first player minimizing objective functions in the Nash sense of the form

$$J_1(u_1,u_2)=\frac{1}{2}x(T)^T E^T K_{1T}Ex(T)$$
$$+\frac{1}{2}\int_0^T \left(x(t)^T Qx(t)+u_1^T(t)R_1u_1(t)-u_2^T(t)R_2u_2(t)\right)dt \tag{4}$$

The second player minimizing the opposite objective function

$$J_2(u_1,u_2)=-J_1(u_1,u_2) \tag{5}$$

with all matrices symmetric. Furthermore Q_i and K_{iT} semi positive definite and R_{ij} positive definite.

3 The Finite Planning horizon

In this section we consider the game (1), (2) under assumption that T is finite. For non-zero-sum game with linear feedback strategy the differential Riccati equation

$$\begin{aligned}&E^T\dot{K}_1+(A-S_2K_2)^TK_1+L_1(A-S_2K_2)\\&-L_1S_1K_1-K_2S_{21}K_2+Q_1=0\end{aligned},$$
$$\begin{aligned}&E^T\dot{K}_2+(A-S_1K_1)^TK_2+L_2(A-S_1K_1)\\&-L_2S_2K_2-K_1S_{12}K_1+Q_2=0\end{aligned},$$
$$L_1E=E^TK_1,$$
$$L_2E=E^TK_2, \tag{6}$$

with

$$S_1=B_1R_{11}^{-1}B_1^T,\ S_2=B_2R_{22}^{-1}B_2^T,$$
$$S_{21}=B_2R_{22}^{-1}R_{12}R_{22}^{-1}B_2^T,\ S_{12}=B_1R_{11}^{-1}R_{21}R_{11}^{-1}B_1^T,$$

play a crucial role. Theorem below give relationship between solution of differential Riccati equation (6) and solution of non-zero-sum linear quadratic game with descriptor system that include feedback strategy.

Theorem 1: The two player linear quadratic differential game with descriptor system (1), (2) has, for every consistent initial state, a linear feedback Nash equilibrium if and only if the set of differential Riccati equation (6) has a set of symmetric solutions K_1,K_2,L_1,L_2 on [0,T].

Proof: Assume

$$u_i^*(t)=F_i^*(t)x(t),\ t\in[0,T],\ i=1,2,$$

is a set of linear feedback equilibrium actions. Then according to the definition of feedback equilibrium, the following linear quadratic regulator problem has a solution $u_1^*(t)=F_1^*(t)x(t)$, for all x_0,
subject to the system

$$E\dot{x}_1=(A+B_2F_2^*(t))^Tx_1(t)+B_1u_1(t),$$
$$Ex(0)=Ex_0.$$

This regulator problem has a solution if the Riccati differential equation

$$E^T\dot{K}_1=(A-B_2F_2^*(t))^TK_1(t)+L_1(t)(A+B_2F_2^*(t))$$
$$-L_1(t)S_1K_1(t)-(Q_1+F_2^{*T}(t)R_{12}F_2^*(t))=0$$
$$L_1(t)E=E^TK_1(t),$$

has a symmetric solution $K_1(.)$ on [0,T]. More over, the solution for this optimization problem is given by

$$u_1^*(t)=-R_{11}^{-1}B_1^TK_1(t)x_1(t).$$

For the second player the proof is analog.

Now we will proof the converse part of the theorem. Assume we choose the feedback strategy

$$F_1=-R_{11}^{-1}B_1^TK_1(t)x(t),\ F_2=-R_{22}^{-1}B_2^TK_2(t)x(t),$$

with $K_1(t)$ and $K_2(t)$ is solution of differential Riccati equation (6). Define

$$\gamma_1(t)=K_1(t)x(t),\ \gamma_2(t)=K_2(t)x(t).$$

Derive the equations to t we have

$$E^T\dot{\gamma}_1(t)=E^T\dot{K}_1(t)x(t)+E^TK_1(t)\dot{x}(t),$$
$$E^T\dot{\gamma}_2(t)=E^T\dot{K}_2(t)x(t)+E^TK_2(t)\dot{x}(t).$$

Based on equation of the system (1) we have

$$E\dot{x}=Ax(t)-B_1R_{11}^{-1}B_1^TK_1(t)x(t)$$
$$-B_2R_{22}^{-1}B_2^TK_2(t)x(t)$$

or

$$E\dot{x}=(A-B_1R_{11}^{-1}B_1^TK_1(t)-B_2R_{22}^{-1}B_2^TK_2(t))x(t).$$

Based on Riccati differential equation (6) we have

$$\begin{aligned}E^T\dot{K}_1&=-(A-S_2K_2)^TK_1-L_1(A-S_2K_2)-Q_1\\&\quad+L_1S_1K_1+K_2S_{21}K_2\\E^T\dot{K}_2&=-(A-S_1K_1)^TK_2-L_2(A-S_1K_1)-Q_2\\&\quad+L_2S_2K_2+K_1S_{12}K_1\end{aligned}.$$

Therefore we have

$$E^T\dot{\gamma}_1(t)=-(A-S_2K_2(t))^TK_1x(t)-Q_1x(t)$$
$$+K_2(t)S_{21}K_2(t)x(t).$$

Based on [10] and the definition of feedback Nash equilibrium it complete the proof. ■

Applying Theorem 1 to zero-sum game descriptor system we will get the following theorem.

Theorem 2: The two player linear quadratic differential game with descriptor system (1), (4), (5) has, for every consistent initial state, a linear feedback Nash equilibrium if and only if the set of differential Riccati equation

$$E^T \dot{K} + A^T K + LA - L(S_1 - S_2)K + Q = 0 \quad (7)$$

has a set of symmetric solutions K, L on [0,T].

Proof: According to Theorem 1, the corresponding generalized differential Riccati equation is

$$E^T \dot{K}_1 + A^T K_1 + L_1 A + Q - L_1 B_1 R_{11}^{-1} B_1^T K_1 - L_1 B_2 R_{22}^{-1} B_2^T K_2 = 0, \quad (8)$$

$$E^T \dot{K}_2 + A^T K_2 + L_2 A - Q - L_2 B_1 R_{11}^{-1} B_1^T K_1 - L_2 B_2 R_{22}^{-1} B_2^T K_2 = 0 \quad (9)$$

Adding (8) and (9) give the following differential equation

$$E^T(\dot{K}_1 + \dot{K}_2) + A^T(K_1 + K_2) + (L_1 + L_2)A - (L_1 + L_2)(B_1 R_{11}^{-1} B_1^T K_1 + L_2 B_2 R_{22}^{-1} B_2^T)(K_1 + K_2) = 0$$

Obviously $(K_1 + K_2)(.) = 0$ and $(L_1 + L_2)(.) = 0$ satisfy this equation. Since the solution of this differential equation is unique we have that $K_1 = -K_2$ and $L_1 = -L_2$. Substitute this into (6) we get (7). ■

4 The Infinite Planning Horizon

In this section we consider the game that the player satisfy (1) and try to minimize the cost function

$$J_i(u_1, u_2) = \frac{1}{2}\int_0^\infty \left(x(t)^T Q_i x(t) + u_1^T(t) R_{i1} u_1(t) + u_2^T R_{i2} u_2(t)\right)dt, \quad (10)$$

with all matrices symmetric. Furthermore Q_i and K_{if} semi positive definite and R_{ij} positive definite.

For the infinite planning horizon case it can be prove that the differential Riccati equation become an algebraic Riccati equation, the solution become constant, and the differential term become zero. Therefore now we consider the algebraic Riccati equation for non-zero-sum game

$$(A - S_2 K_2)^T K_1 + L_1(A - S_2 K_2) - L_1 S_1 K_1 - K_2 S_{21} K_2 + Q_1 = 0,$$

$$(A - S_1 K_1)^T K_2 + L_2(A - S_1 K_1) - L_2 S_2 K_2 - K_1 S_{12} K_1 + Q_2 = 0,$$

$$L_1 E = E^T K_1,$$

$$L_2 E = E^T K_2, \quad (11)$$

with

$$S_1 = B_1 R_{11}^{-1} B_1^T, S_2 = B_2 R_{22}^{-1} B_2^T,$$

$$S_{21} = B_2 R_{22}^{-1} R_{12} R_{22}^{-1} B_2^T, \; S_{12} = B_1 R_{11}^{-1} R_{21} R_{11}^{-1} B_1^T.$$

For zero-sum linear quadratic dynamic game with infinite planning horizon we have the corresponding generalized algebraic Riccati equation

$$A^T K + LA + Q - L(B_1 R_1^{-1} B_1^T + B_2 R_2^{-1} B_2^T)K = 0 \quad (12)$$

with $LE = E^T K$.

Theorem 1 and Theorem 2 also can be applied for infinite horizon case. Based on generalized eigen value define in [15] and [22] we will derive solution of the algebraic Riccati equation by consider invariant subspace that we derive from the generalized eigen vector.

5 Solution With Invariant Subspace Method

Hamiltonian matrix that correspond with algebraic Riccati equation (12) will satisfy

$$\tilde{A} = \begin{pmatrix} A & -S_1 + S_2 \\ -Q & -A^T \end{pmatrix}.$$

Consider descriptor system

$$\tilde{E}_b \dot{x} = \tilde{A} x \quad (13)$$

with

$$\tilde{A} = \begin{pmatrix} A & -S_1 + S_2 \\ -Q & -A^T \end{pmatrix}, \quad \tilde{E}_b = \begin{pmatrix} E & 0 \\ 0 & E^T \end{pmatrix}.$$

In [5, 22] it is defined the generalized eigenvalue problem correspond with (13) that satisfy

$$\tilde{A}\lambda = \lambda \tilde{E}_b z, \quad (14)$$

with

$$rank\tilde{E} = 2r < 2n,$$

if assume that $rankE = r < n$.

Every descriptor system is equivalent with descriptor system such that the left hand matrix coefficient in descriptor system is

$$\tilde{E} = \begin{pmatrix} I_{2r} & 0 \\ 0 & 0 \end{pmatrix}.$$

Therefore it is enough to consider the descriptor system with this matrix $\tilde{E}$ and $\tilde{A}$ that satisfy (13).

Below are results for invariant subspace of the generalized eigen value problem (14) that will be used to get the optimal Nash solution for the non-cooperative game.

Lemma 1: If generalized eigen vectors in equation (14) are e_i with i corresponde with non-zero diagonal element of $\tilde{E}$ then the subspace generate by the eigen vector will form invariant subspace to $\tilde{A}$.

Proof: The generalized eigen vector will satisfy

$$\tilde{A}(ax) = a\tilde{A}x = a\lambda\tilde{E}x = a\lambda x . \blacksquare$$

The eigen vector that satisfy Lemma 1. involved the finite eigen value (Katayama et al, 1992). Based on Katayama (1992), solutions of the generalized eigen value problem, will consist of finite eigen value and infinite eigen value.

Lemma 2: The infinite eigen value will generate the $\tilde{A}$ invariant subspace.

Proof: Take the vector x which is infinite eigen value, then we get $\tilde{E}x = 0$. We get

$$\tilde{A}(ax) = a\tilde{A}x = a\lambda\tilde{E}x = 0 .$$

It give proof that the eigen vector generate the $\tilde{A}$ invariant subspace. ■

For the generalized finite eigen value that generate the invariant subspace we get

$$\tilde{A}U = \tilde{E}UW .$$

Partisize U as

$$U = \begin{pmatrix} U_1 & U_2 \end{pmatrix}. \tag{15}$$

If

$$rank\tilde{E} = 2r \le 2n$$

Then $\tilde{E}$ will have *(2n-2r)* zero eigen value. Let N_1 is eigen vector that correspond to the zero eigen value. We get

$$\tilde{E}N_1 = 0 . \tag{16}$$

Let N_2 is matrix that correspond with the *(n-r)* zero eigen value of $\tilde{E}^T$. We get

$$\tilde{E}^T N_2 = 0 \quad . \tag{17}$$

Let the set of $\tilde{A}$-invariant subspaces is given by

$$\tilde{A}^{inv} = \left\{ T \middle| \tilde{A}T \subset T \right\}.$$

The set of graph subspaces plays a crucial role that given by

$$K^{pos} := \left\{ K \in \tilde{A}^{inv} \middle| K \oplus \mathrm{Im}\begin{pmatrix} 0 \\ I \end{pmatrix} = \Re^{2n} \right\}.$$

Every element of K^{pos} defines one solution of algebraic Riccati equation [10].

Based on theory on descriptor system [15, 22] for the generalized eigen value problem with

rank(E)=r

will exist r finite eigen value and n-r infinite eigen value. Based on Lemma 2 the infinite eigen vector will generate invariant subspace. We can summarize that for the non-cooperative game problem that satisfy (1), (4) and (5) will exist 2r finite eigen value and 2*(n-r)* infinite eigen value. Hence based on Lemma 1 and Lemma 2 will exist as much as possible 2n eigen vector that generate invariant subspace.

It has been prove that if all eigen value have algebraic multiplicity one, for ordinary system, will exist

$$\begin{pmatrix} 2n \\ n \end{pmatrix}$$

optimal Nash solution (that compiled by the eigen vectors) [10, 12]. It can be summarized that for the dynamic game with descriptor system if all eigen value have algebraic multiplicity one, will exist

$$\begin{pmatrix} 2r \\ r \end{pmatrix}$$

optimal Nash solution that compiled by the finite eigen value.

Theorem 3: If $K = YX^{-1}$ for $\mathrm{K} := \mathrm{Im}\begin{pmatrix} X \\ Y \end{pmatrix} \in \mathrm{K}^{pos}$, then it will solve ARE.

Proof: Let K solution of ARE (2.3.17). Then we have

$$\tilde{A}\begin{pmatrix} I \\ K \end{pmatrix} = \tilde{A}\begin{pmatrix} I \\ K \end{pmatrix} J = \begin{pmatrix} E & 0 \\ 0 & E^T \end{pmatrix}\begin{pmatrix} I \\ K \end{pmatrix} J$$

$$= \begin{pmatrix} EJ \\ E^T KJ \end{pmatrix} = \begin{pmatrix} EJ \\ LEJ \end{pmatrix}$$

Substitute $\tilde{A}$ we get

$$\begin{pmatrix} A & -S_1 + S_2 \\ -Q & -A^T \end{pmatrix}\begin{pmatrix} I \\ K \end{pmatrix}$$

$$= \begin{pmatrix} EJ \\ LEJ \end{pmatrix}\begin{pmatrix} EJ \\ L_1 EJ \\ L_2 EJ \end{pmatrix}$$

From the first row we get

$$EJ = A - S_1 K + S_2 K\,.$$

From second row we get

$$-Q - A^T K = LEJ = LA - LS_1 K + LS_2 K\,.$$

Or we get

$$-Q - A^T K - LA + L(S_1 - S_2)K = 0$$

That is mean that ARE is satisfied. ■

We will have that $(U_1 \quad N_1)$ is invertible. Therefore we get the formulae to get the optimal Nash solution for the game (see [15] and [22] for details).

Theorem 4: Let

$$\begin{pmatrix} X \\ Y \end{pmatrix} = \begin{pmatrix} U_1 & N_1 \\ U_2 & N_2 \end{pmatrix}.$$

Then the formulae for the optimal gain is

$$K = (U_2 \quad N_2)(U_1 \quad N_1)^{-1}. \tag{18}$$

The optimal Nash solution are given by

$$u_1(k) = -R_1^{-1} B_1^T K x(k+1),$$
$$u_2(k) = -R_2^{-1} B_2^T K x(k+1)$$

6 Numerical Example

Consider game with the players give control to system

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} 1 \\ 1 \end{pmatrix} u_1 + \begin{pmatrix} 0 \\ 1 \end{pmatrix} u_2,$$

$$x_1(0) = x_{10}, \quad x_2(0) = x_{20}\ .$$

Let the matrices in the cost function (10) are

$$Q = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad R_1 = [1], \quad R_2 = [-1].$$

Consider eigen value problem (13) with

$$\tilde{A} = \begin{pmatrix} A & -S_1 + S_2 \\ -Q & -A^T \end{pmatrix}, \quad \tilde{E} = \begin{pmatrix} E & 0 \\ 0 & E^T \end{pmatrix}.$$

Or

$$\tilde{A} = \begin{bmatrix} 0 & 1 & -2 & -1 \\ 1 & 1 & -1 & -1 \\ -1 & 0 & 0 & -1 \\ 0 & -1 & -1 & -1 \end{bmatrix}, \tilde{E} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

We get

$$K_1 = (0 \quad 1), \quad K_2 = (0.6667 \quad 0.1111)$$

The controlled system is

$$\begin{pmatrix} 0.6667 & 1.1111 \\ 1 & -1 \end{pmatrix}.$$

The system response is simulated with MATLAB and given in Figure 1 and Figure 2. While the control of first player and second player are given in Figure 3 and Figure 4. The simulation show that the feedback control can regulate the state.

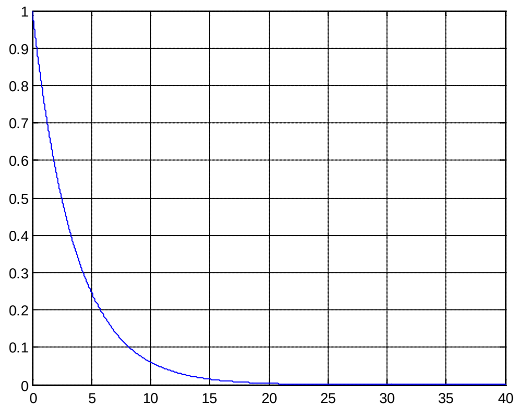

Figure 1. System response of x_1

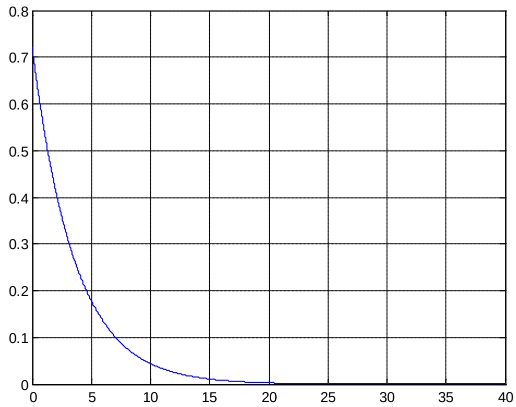

Figure 2. System response of x_2

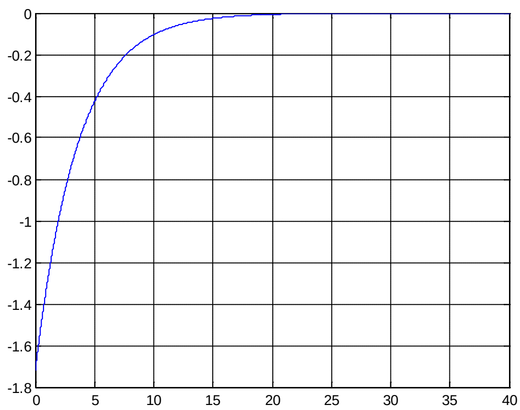

Figure 3. Control of first player u_1

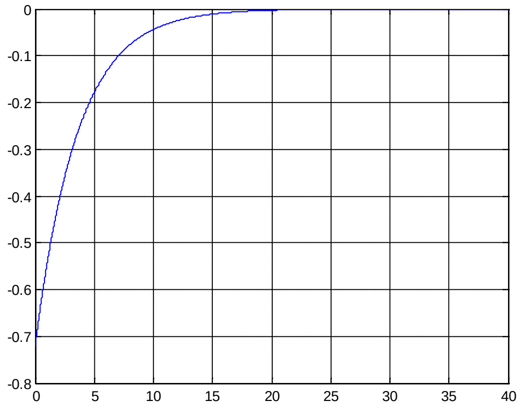

Figure 4. Control of second player u_2

7 Conclusion

This paper consider Nash equilibrium of linear quadratic dynamic game for descriptor systems for finite horizon and infinite horizon case in linear feedback scenario for zero sum case. The paper consider Riccati-type differential equation for finite horizon case and algebraic Riccati-type equation for infinite horizon case. We derive theorem that consider relationship between solution of the Riccati equation and solution of the game. Moreover we give formulae to find the feedback Nash equilibrium with invariant subspace method.

References:

[1] Basar, T., P Bernhard, *Optimal control and related Minimax Design problems*, Modern Birkhauser Classics, 1995
[2] Basar T., and Olsder, G.J., *Dynamic Noncooperative Game Theory*, second Edition, Academic Press, 1995.
[3] Brenan, K.E., Campbell, S.L., and, Petzold, L.R., *Numerical Solution of Initial Value Problems in Differential-Algebraic Equations*, Classics in Applied Mathematics, Vol. 14, SIAM Philadelpia, 1996.
[4] Broek, W. A. van den, Engwerda, J. C. and Schumacher, J. M., Robust Equilibria in Indefinite Linear-Quadratic Differential Games, *JOTA*, vol. 119, 2003, pp. 565-595.
[5] Byers, R., Geerts, T., and Mehrmann, V., “Descriptor systems without controllability at infinity “, *SIAM J. Control Optim*, Vol. 35, No.2, 1997, pp 462-479
[6] Cobb, D., “Controllability, Observability and Duality in Singular Systems”, *IEEE Transactions on Automatic Control*, vol AC-29, no.12, 1984, 1076-1082.
[7] Dai, L., *Singular Control Systems*, Springer Verlag, Berlin, 1989.
[8] Engwerda, J.C., Salmah, The Open-Loop Linear Quadratic Differential Game for index one Descriptor Systems, *Automatica*, Vol. 45, No. 2, 2009, pp, 585-592.
[9] Engwerda, J.C., Uniqueness conditions for the affine open-loop linear quadratic differential game, *Automatica*, 44(2),2007, pp 504-511.
[10] Engwerda, J.C., *LQ Dynamic Optimization and Differential Games*. Chichester: John Wiley & Sons, 2005.
[11] Engwerda, J.C., Salmah, “Feedback Nash Equilibria for Linear Quadratic Descriptor Differential Games”, *Automatica*, vol. 48, pp 625-631, 2012.
[12] Freiling, G., Jank, G., , Abou-Kandil, Discrete Time Riccati Equations in Open-Loop Nash and Stackelberg Game, *European Journal of Control*, 5, 1999, pp 56-59.
[13] Gantmacher, *Theory of Matrices*, Vol.1, Chelsea Publishing Company, New York, 1959.
[14] Gerstner, A.B, Mehrmann, V, and Nichols, N.K, “Regularization of descriptors systems by derivative and proportional state feedback”, *SIAM J. Matrix Anal. Appl*, Vol.13, No.1, 1992, 46-47.
[15] Katayama, T., and Minamino, K., “Linear Quadratic Regular and Spectral Factorization for Continuous Time Descriptor Systems”, *Proceedings of the 31st Conference on decision and Control*, Tucson, Arizona, 1992, 967-972.
[16] Kautsky, J., Nicholas, N.K., Chu, E. K. W., “Robust pole assignment in singular control systems”, *Linear Algebra and its Applications*, 121, 9-37, 1989.
[17] Kumar, A, Daoutidis, P., State-space Realizations of Linear Differential Algebraic Equation Systems with Control-dependent State Space, *IEEE Transactions on Automatic Control*, Vol; 41, 1996, 269-274.
[18] Lewis, F.L., “A survey of Linear Singular Systems”, *Circuits System Signal Process* vol.5, no.1, 1986, 3-36.
[19] Luenberger, D. G., Dynamic Equation in Descriptor Form, *IEEE Transaction on Automatic Control*, vol. 22, pp. 312-321, 1977.
[20] Luenberger, D. G. and Arbel, Singular dynamic Leontief systems, *Econometrica*, vol. 45, pp. 991-995, 1977
[21] Mehrmann, V.L, *The Autonomous Linear Quadratic Control Problem. In: Lecture Notes in Control and Information Sciences (Eds. M.*

Thoma and A. Wyner), Vol.163. Berlin: Springer, 1991.

[22] Minamino, K., *The Linear Quadratic Optimal Regulator and Spectral Factorization for Descriptor Systems*, Master Thesis, Department of Applied Mathematics and Physics, Faculty of Engineering, Kyoto University, 1992.

[23] Musthofa, M W, Salmah, Engwerda, J.C., Suparwanto, A., "The open-loop zero-sum linear quadratic impulse free descriptor differential game", *International Journal on Applied mathematics and Statistics*, Vol. 35, No. 5, 2012, pp 29-44.

[24] Salmah, *Kontrol Optimal Sistem Regulator Deskriptor Untuk Permainan Dinamis (Optimal Control Regulator Descriptor For Dynamic Game)*, Dissertation, Universitas Gadjah Mada, Yogyakarta, Indonesia, 2006.

[25] Xu, H., and Mizukami, K., Two-person Two-criteria Decision Making Problem for Descriptor Systems, *JOTA*, vol. 78, pp. 1993, 163-173.

[26] Xu, H., and Mizukami, K., On The Isaacs Equation of Differential Game for Descriptor Systems, *JOTA*, vol. 83, 1994, pp. 405-419.