A family of adaptive $H_\infty$ controllers for switched dissipative Hamiltonian systems

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Abstract: This paper investigates adaptive $H_\infty$ control problem for switched dissipative Hamiltonian systems (SDHSs). Firstly, using the dissipative Hamiltonian structural properties, such systems generated an augmented SDHS, with which some results on the controller design are then obtained. When there are external disturbances and parameter perturbations in such systems, a family of adaptive $H_\infty$ controller is designed using the multiple Lyapunov functions (MLFs). Secondly, an algorithm for solving parameters of controllers is proposed with symbolic computation. The proposed method avoids solving Hamilton-Jacobi-Issacs inequalities (equations), and the obtained controllers are relatively simple in form and easy in operation. Finally, a numerical example is studied by using the new results proposed in this paper to SDHSs with external disturbances and parameter perturbations, and some numerical simulations are carried out to support our new results.

Key Words: adaptive $H_\infty$ control; switched dissipative Hamiltonian systems (SDHSs); multiple Lyapunov functions (MLFs); symbolic computation.

1 Introduction

Switched systems are an important class of hybrid systems, which have developed rapidly in recent years. The motivation for studying switched systems is from the fact that many practical systems are inherently multi-models in the sense that several dynamical subsystems are required to describe their behaviour depending on various changing environmental factors. The widespread applications of switched systems are also motivated by increasing performance requirements in control, especially in the presence of large disturbances or uncertainties [1-4]. A common Lyapunov function for all subsystems is a necessary and sufficient condition for switched system to be asymptotically stable under arbitrary switching laws [5, 6]. It has been shown that the multiple Lyapunov functions (MLFs) approach proposed in [7] is an effective tool for choosing such switching laws. A hybrid nonlinear control methodology for a broad class of switched nonlinear systems with input constraints was proposed [8], which is the integrated synthesis via the MLFs. The sufficient conditions for exponential stability and weighted $L_2$-gain developed for a class of switching signals with average dwell time [9]. The MLFs is derived a sufficient condition for the switched nonlinear system to be asymptotically stable with $H_\infty$-norm bound [10]. The switched dissipative Hamiltonian system (SDHS) is a kind of important nonlinear hybrid systems. Such system not only plays an important role in development of hybrid control theory, but also finds many applications in practical control designs for obtaining better control performance [11, 12]. The stability of SDHSs under arbitrary switching paths has been investigated via the dissipative Hamiltonian structural properties and the MLFs [13].

Adaptive control problems and the parameterization of all stabilizing controllers are two of the most researched and written about issues in modern control. Adaptive $H_\infty$ control has proved to be a very useful method in nonlinear control systems which are sensitive to parameter uncertainty. With adaptive $H_\infty$ control it is possible to estimate parameters errors and to compensate for those errors, and improve robustness. Some basic adaptive control methods for first-order, linear, nonlinear, single-input and multi-input systems have been described [14]. [15] described recursive methodology of back-stepping for nonlinear and adaptive control design. In [16], a novel approach for the solution of nonlinear adaptive control
problems is proposed. The approach for the solution of nonlinear adaptive control problems is proposed. Furthermore, a family of controllers can solve the underlying \( H_\infty \) control problems for a controller satisfying the additional design objectives, which are internal stability and disturbance attention. A class of local nonlinear \( H_\infty \) controllers is parameterized as nonlinear fractional transformations on contractive, stable nonlinear parameters via output feedback have proposed [17,18]. In [19,20], the state-space formulas extended and a family of \( H_\infty \) state-feedback controller presented for n-dimensional nonlinear system. A family of reliable nonlinear \( H_\infty \) controller has been proposed via solving the HJ equations [21]. The generalized Hamiltonian system studied and a family of parameterized controller proposed in \( H_\infty \) control and adaptive control [22,23]. The controllers [17-23] are intended to solve the control problem for just one system. Particularly, there are few results on adaptive \( H_\infty \) control design of SDHSs so far.

Therefore, how to find a method for designing parameterized controller to solve adaptive \( H_\infty \) problem for SDHSs is a challenging issue. In this paper, we show that all the Hamiltonian function of the subsystems can be used as the MLFs for the SDHSs under a realistic assumption. We present a novel, straightforward and convenient method to design a family of controllers to insure that the SDHSs are adaptive \( H_\infty \) control.

The remainder of this paper is organized as follows. In Section 2, the problem of adaptive \( H_\infty \) control for SDHSs is formulated. The main contributions of this paper are then given in Section 3, in which a family of controllers and an algorithm for solving parameters of controller are provided, respectively. We present a numerical example for illustrating effectiveness and feasibility of controller in Section 4 and conclusions follow in Section 5.

2 Problem Formulation

Consider the following SDHSs with disturbances and parameter perturbations

\[
\dot{x} = [J_{\lambda(t)}(x,p) - R_{\lambda(t)}(x,p)]\nabla H_{\lambda(t)}(x,p) + g_{\lambda(t)}(x)u + \overline{g}_{\lambda(t)}(x)\omega \\
z = h_{\lambda(t)}(x)g_{\lambda(t)}^T(x)\nabla H_{\lambda(t)}(x) \\
x(t_0) = x_0
\]

(1)

where \( x \in \mathbb{R}^n \) is the state vector; \( u \in \mathbb{R}^m \) is the controller; \( \omega \in \mathbb{R}^r \) is the disturbances; \( p \in \mathbb{R} \) is a constant unknown vector representing parameter perturbations; \( J_{\lambda(t)}^T(x,p) = -J_{\lambda(t)}(x,p) \in \mathbb{R}^{m \times n} \), \( J_{\lambda(t)}^T(x,0) = J_{\lambda(t)}(x) \); \( 0 \leq R_{\lambda(t)}(x,p) \in \mathbb{R}^{m \times m} \), \( R_{\lambda(t)}(x,0) = R_{\lambda(t)}(x) \); \( g_{\lambda(t)}(x) \in \mathbb{R}^{m \times m} \) and \( \overline{g}_{\lambda(t)}(x) \in \mathbb{R}^{n \times n} \) are sufficiently smooth functions; \( h_{\lambda(t)}(x) \) is a weighting matrix; \( z \in \mathbb{R}^q \) is the penalty; \( H_{\lambda(t)},(x,p) \) is the subsystem’s Hamiltonian function (the total energy) satisfying \( H_{\lambda(t)}(x,p) > 0 \), \( H_{\lambda(t)}(x,0) = H_{\lambda(t)}(x) \) and

\[
\nabla H_{\lambda(t)}(x,p) = \frac{\partial H}{\partial x}(x,p) \quad ; \quad \text{the map} \\
\lambda(t):[t_0, +\infty) \to \Lambda = \{1,2,\cdots,N\} \text{ is a piecewise constant one, called the switching law or switching path, and} \\
\lambda(t) = i (i = 1,2,\cdots,N) \text{ denotes that the} \quad i \text{ th subsystem is realized. For an arbitrary} \\
\text{switching law} \lambda(t), \{t_m\}_{m=0}^{\infty} \text{ is called the switching time sequence, which is assumed to satisfy} \\
t_0 < t_1 < t_2 < \cdots < t_m < \cdots + \infty. \text{ If} \quad t_{m+1} = \infty, \text{ the} \quad i \text{ th} \\
\text{subsystem of system (1) is always realized in} \quad [t_m, +\infty) \text{ and the whole system is naturally stable.} \\
\text{To use the dissipative Hamiltonian structural properties and the MLFs, we propose two} \quad \text{assumptions as follows.}
\]

Assumption 1. For \( \forall i \in \Lambda \), the Hamiltonian function \( H_i(x) \) satisfies \( H_i(x) \in C^2 \) and the Hessian matrix \( \text{Hess}(H_i(x)) > 0 \).

Remark 1. Note that \( H_i(x) \) has a local minimum at the equilibrium \( x_0 \) of system (1). It is straightforward that in Assumption 1, \( H_i(x) \in C^2 \) guarantees the existence of \( \text{Hess}(H_i(x)) \) and \( \text{Hess}(H_i(x)) > 0 \) guarantees that \( H_i(x) \) is strict convex on some neighborhood of equilibrium \( x_0 \).

Assumption 2. For \( \forall i \in \Lambda \) and \( \forall x, y \in \mathbb{R}^n \), the Hamiltonian function \( H_i(x) \) satisfies

\[
H_i(x) > H_i(y) \quad \Leftrightarrow \quad \|x\|_p > \|y\|_p, \quad \text{where} \\
\|x\|_p \triangleq \sup x^TPx \text{ and} \quad P > 0 \text{ is a positive definite matrix.}
\]

Remark 2. Assumption 2 implies that the Hamiltonian function (candidate Lyapunov function)
$H_i(x)$ increases with the increase of $\|x\|_p$. Obviously, this assumption can be satisfied for many Hamiltonian systems. Thus, Assumption 2 is a realistic one.

**Lemma 1.** [13] If Assumption 2 holds, $u \equiv 0$, $\omega \equiv 0$ and $R_i(x) > 0$, $\forall \in \Lambda$, then system (1) is globally asymptotically stable under the arbitrary switching laws $\lambda(t)$ with its dwell time $\tau > 0$.

**Definition 1.** The problem considered in this paper is to propose a family of adaptive $H_\infty$ controllers for systems (1), which can be described as: given a disturbance attenuation level $\gamma > 0$ and an arbitrary switching law $\lambda(t)$, we can obtain a family of adaptive $H_\infty$ controller of the form

$$
\begin{align*}
    u &= u_{\alpha(t)}(x, \hat{\theta}) \\
    \dot{\hat{\theta}} &= \beta_{\alpha(t)}(x, \hat{\theta})
\end{align*}
$$

such that

**R1:** For $\forall i \in \Lambda \ \forall \omega \ , \ t \in T_m$ and $m \in Z_+$, the inequality $\dot{H}_i(x) + \Psi_i(x) \leq \frac{1}{2} \left\{ \gamma^2 \|\omega\|^2 - \|\tilde{g}\|^2 \right\}$ holds along the trajectories of the closed-loop system, which consisted of system (1) and the controller (2), where $\Psi_i(x) \geq 0$ is a scalar function.

**R2:** The closed-loop system (1) is asymptotically stable when $\omega \equiv 0$.

### 3 Main Results

In this section, we propose a family of adaptive $H_\infty$ controller for system (1) by using the dissipative Hamiltonian structural properties and the MLFs and an algorithm for solving parameters with symbolic computation. The parameterization method suggests a framework to solve adaptive $H_\infty$ control problem of SDHs.

**Notation 1.** $\nabla H_i = \nabla H_i(x)$, $\Phi_i = \Phi_i(x)$, $g_i = g_i(x)$, $\tilde{g}_i = \tilde{g}_i(x)$, $h_i = h_i(x)$, $K_i = K_i(x)$.

#### 3.1 A family of adaptive $H_\infty$ controllers

**Assumption 3.** There exits an $m \times n_i$ matrix $\Phi$ such that

$$
\begin{align*}
    [J_i(x, p) - R_i(x, p)] \Delta H_i(x, p) &= g_i \Phi_i \dot{\theta}_i
\end{align*}
$$

where $\Delta H_i(x, p) = \nabla H_i(x, p) - \nabla H_i$ and $\theta_i \in \mathbb{R}^n$ is a constant parameter perturbation vector depending on $p$.

**Remark 3.** Assumption 3 is the matched condition, a common assumption in adaptive control of Hamiltonian systems. In most cases, we can find $\Phi_i$ and $\theta_i$ such that (3) holds [24].

**Theorem 1.** Suppose Assumption 1 and 2 hold for system (1) and give a disturbance attenuation level $\gamma > 0$, $\forall \in \Lambda$

$$
\begin{align*}
    R_i(x, p) + \frac{1}{2\gamma^2} \left( g_i g_i^T - \tilde{g}_i \tilde{g}_i^T \right) &\geq 0 \\
    \nabla H_i^T g_i K_i &\leq 0
\end{align*}
$$

hold simultaneously. Then under an arbitrary switching law $\lambda(t)$, adaptive $H_\infty$ control of system (1) can be realized by following controller which satisfies the rules R1 and R2.

$$
\begin{align*}
    u &= u_{\alpha(t)}(x, \hat{\theta}) = -\frac{1}{2} \left( g_i h_i^T + \frac{1}{\gamma^2} I_m \right) g_i^T \nabla H_i + K_i - \Phi_i \dot{\hat{\theta}} \\
    \dot{\hat{\theta}} &= Q_i \phi_i^T g_i^T \nabla H_i
\end{align*}
$$

where $K_i \in \mathbb{R}^{m \times 1}$ is parameterized part of controller, $I_m$ is an $m \times m$ unit matrix, $\hat{\theta}_i$ is an estimation of $\theta_i$ and $Q_i$ is the adaptation gain matrix, which is constant, and note

$$
\Psi_i(x) = \nabla H_i^T \left( R_i(x, p) + \frac{1}{2\gamma^2} \left( g_i g_i^T - \tilde{g}_i \tilde{g}_i^T \right) \right) \nabla H_i.
$$

**Proof.** Suppose $\lambda(t) = i \in \Lambda , \ t \in T_m , \ m \in Z_+$ is an arbitrary switching law. Substituting (6) into (1) and using condition (3) yield

$$
\begin{align*}
    \dot{x} &= \left[ J_i(x, p) - R_i(x, p) \right] \nabla H_i - \frac{1}{2} g_i h_i^T h_i g_i^T \nabla H_i \\
    &\quad - \frac{1}{2\gamma^2} g_i g_i^T \nabla H_i + g_i K_i + g_i \Phi_i \left( \theta - \hat{\theta} \right) + \tilde{g}_i \omega \\
    \dot{\hat{\theta}} &= Q_i \phi_i^T g_i^T \nabla H_i \\
    \dot{z} &= h_i g_i^T \nabla H_i
\end{align*}
$$

which can be rewritten as an augmented Hamiltonian system.
\[
\dot{X} = \left[ J_i(X) - \bar{R}_i(X) \right] \nabla H_i(X) + G_i \nu_i + \bar{G}_i(X) \omega \\
z = h_i g_i^T \nabla H_i(X)
\]

(8)

where \( X = \left[ x^T, \dot{\theta}^T \right]^T \), \( G_i(X) = \left[ g_i^T, 0 \right]^T \), \( \bar{G}_i(X) = \left[ \bar{g}_i^T, 0 \right]^T \), \( \nu_i = [K_i, 0]^T \) are with proper dimensions; \( \nabla H_i(X) = \left[ \nabla H_i^T, \nabla H_i(\dot{\theta})^T \right]^T \),

\[
J_i(X) = \left[ J_i(x, p) - (Q \Phi_i^T g_i^T)^T \right]
\]

\[
R_i(X) = \begin{bmatrix}
R_i(x, p) + \frac{1}{2} g_i h_i^T h_i g_i^T + \frac{1}{2} g_i g_i^T  & 0 \\
Q \Phi_i^T g_i^T & 0
\end{bmatrix}
\]

\[
\bar{R}_i(X) = H_i + \frac{1}{2} (\theta - \dot{\theta})^T Q^{-1} (\theta - \dot{\theta}).
\]

It is easy see that system (8) is still a class of dissipative Hamiltonian systems.

We choose the candidate Lyapunov function \( V_i(X) = \bar{H}_i(X) - c \geq 0 \) (c is some number such that \( V_i(X) \) is positive definite near \( X_0 = (x_0^T, \theta_0^T)^T \) and \( \bar{H}_i(X) \) has a local minimum at \( X_0 \)). For system (8), we have

\[
\bar{H}_i(X) = \nabla \bar{H}_i^T(X) \left[ J_i(X) - \bar{R}_i(X) \right] \nabla \bar{H}_i(X)
\]

\[
+ \nabla \bar{H}_i^T(X) G_i \nu_i + \nabla \bar{H}_i^T(X) \bar{G}_i \omega
\]

\[
= -\nabla \bar{H}_i^T \left( R_i(x, p) + \frac{1}{2} g_i h_i^T h_i g_i^T + \frac{1}{2} g_i g_i^T \right) \nabla H_i
\]

\[
+ \nabla H_i^T g_i K_i + \nabla H_i^T \bar{g}_i \omega
\]

\[
- \frac{1}{2} \nabla H_i^T g_i h_i^T h_i g_i^T \nabla H_i + \nabla H_i^T \bar{g}_i \omega
\]

\[
= -\nabla \bar{H}_i^T \left( R_i(x, p) + \frac{1}{2} g_i h_i^T h_i g_i^T + \frac{1}{2} g_i g_i^T \right) \nabla H_i
\]

\[
+ \nabla H_i^T g_i K_i + \frac{1}{2} \left( \gamma^2 \| \phi_i^T - \| \phi_i^T \| \right)
\]

\[
- \frac{1}{2} \gamma \left\| \bar{g}_i \right\|^2 \nabla H_i - \gamma \omega_i.
\]

(9a)

We obtain

\[
\bar{H}_i(X) + \nabla \bar{H}_i^T \left( R_i(x, p) + \frac{1}{2} g_i h_i^T h_i g_i^T + \frac{1}{2} g_i g_i^T \right) \nabla H_i - \nabla H_i^T g_i K_i
\]

\[
= \bar{H}_i(X) + \Psi_i(X)
\]

\[
= \frac{1}{2} \left( \gamma^2 \| \phi_i^T - \| \phi_i^T \| \right) - \frac{1}{2} \gamma \left\| \bar{g}_i \right\|^2 \nabla H_i - \gamma \omega_i
\]

\[
\leq \frac{1}{2} \left( \gamma^2 \| \phi_i^T - \| \phi_i^T \| \right)
\]

(9b)

where \( \Psi_i(X) = \bar{R}_i(X) + \frac{1}{2} \gamma \left( G_i G_i^T - \bar{G}_i G_i^T \right) - G_i \nu_i \geq 0 \)

is a scalar function.

So the rule R1 can be satisfied, which implies the \( L_2 \) gain of the closed-loop system (8) controlled by controller (6) (from \( \omega \) to \( z \)) is bounded by \( \gamma \).

Next, we prove that the closed-loop system is asymptotically stable at \( x_0 \) under the arbitrary switching laws \( \lambda(t) \), when \( \omega = 0 \).

When \( \omega = 0 \), the closed-loop system can be expressed

\[
\bar{H}_i(X) = \nabla \bar{H}_i^T(X) \left[ J_i(X) - \bar{R}_i(X) \right] \nabla \bar{H}_i(X)
\]

\[
+ \nabla \bar{H}_i^T(X) G_i \nu_i
\]

\[
= -\nabla \bar{H}_i^T \left( R_i(x, p) + \frac{1}{2} g_i h_i^T h_i g_i^T + \frac{1}{2} g_i g_i^T \right) \nabla H_i
\]

\[
+ \nabla H_i^T g_i K_i + \nabla H_i^T \bar{g}_i \omega
\]

\[
= -\nabla \bar{H}_i^T \left( R_i(x, p) + \frac{1}{2} g_i h_i^T h_i g_i^T + \frac{1}{2} g_i g_i^T \right) \nabla H_i
\]

\[
+ \nabla H_i^T g_i K_i + \frac{1}{2} \left( \gamma^2 \| \phi_i^T - \| \phi_i^T \| \right)
\]

\[
- \frac{1}{2} \gamma \left\| \bar{g}_i \right\|^2 \nabla H_i - \gamma \omega_i
\]

(10)

From Lemma 1, the rule R2 can be satisfied.

So the closed-loop system (1) controlled by controller (6) is asymptotically stable under the arbitrary switching laws \( \lambda(t) \). This completes the proof. □

Remark 4. (1) The condition (4) in Theorem is not restrictive, and can be easily satisfied in many systems.

(2) \( K_i \) is a polynomial vector with parameters.

We can obtain the parameters of \( K_i \) via solving condition (5).
(3) The proposed parameterization method can be used for general nonlinear switched systems, and of course the first step in applying the method is to express the nonlinear systems as DSHSs based on dissipative Hamiltonian realization methods [25, 26].

3.2 Solving parameters algorithm (SP)

From condition (4), we can obtain the $\gamma_1^*$. Let $\gamma \geq \max \{\gamma_1^*\}$ such that condition (4) holds. Then we propose an algorithm to find parameters ranges of controller (6) via solving the parameters of $K_i$ in condition (5), respectively. Note $N(x) = K_i$. The SP algorithm now proceeds as follows.

S1. Set $N(x) = [N_1(x) N_2(x) \cdots N_m(x)]^T$ and suppose a positive integer $r$, which is the degree of polynomial vector $N(x)$. Write $N_i(x) = \sum_{i=1}^{\infty} a_{ri} p_r(x)$, where $l = \sum c(n + r - 1, r)$, $p_r(x) = \prod_{i=1}^{n} x_i$ and $n$ is the number of state variable.

S2. Let $S = -\nabla H_{p}^T g(N(x))$.

S3. The influence of high order items can be ignored because this paper consider locally asymptotically stable for system. Choose all terms of $\text{deg}(S) \geq 3$ and $\text{deg}(S) = 1$ from $S$ and let the coefficients of these terms be zero. So obtain a set of equations $A$.

S4. Obtain a set of parameters solution $U_i$ via solving $A$ by using cylindrical algebraic decompositions (CAD) algorithm [27].

S5. Substitute $U_i$ into $S$ and obtain a new polynomial $S'$, which is a quadratic form.

S6. Rewrite $S'$ as coefficient matrix $M$, and all principal minors of $M$ must be positive semi-definite [28]. Choose all principal minors of $M$ and obtain inequalities $B$.

S7. Obtain a set of parameters solution $U_2$ via solving $B$ by using CAD algorithm.

S8. Let $U = U_1 \cup U_2$ and substitute $U$ into controller (4), thus obtain the polynomial parameterized controller. This completes the algorithm. \( \square \)

Remark 5. (1) The SP algorithm starts from $r = 1$ normally.

(2) The CAD algorithm is given by Semi-Algebraic-Set-Tools of Regular-Chains in Maple 16.

(3) The set of parameters solution obtained by SP algorithm is a subset of solutions of results by using CAD algorithm.

4 Numerical experiment

Consider a SDHS with disturbances and parameter perturbations,

$$J_1(x, p) = \begin{bmatrix} 0 & x_1^2 - \sin(x_2 x_3) & -1 \\ - x_1^2 + \sin(x_2 x_3) & 0 & 2 \\ 1 & -2 & 0 \end{bmatrix}$$

$g_{11} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 1 \end{bmatrix}$, $g_{12} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $R_1(x, p) = \text{Diag}\{1, 2, 1\}$,

$H_1(x, p) = x_1^2 + x_2^2 + (1 + p_1)x_3^2$,

$$J_2(x, p) = \begin{bmatrix} 0 & -x_1^2 & -2 \\ -x_1^2 & 0 & -1 \\ 2 & 1 & 0 \end{bmatrix}$$

$g_{21} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \\ 1 & 0 \end{bmatrix}$, $g_{22} = \begin{bmatrix} 0 & 2 \\ 1 & 0 \end{bmatrix}$, $R_2(x, p) = \text{Diag}\{1, 3, 2\}$

$H_2(x, p) = \frac{1}{2} x_1^2 + 2 x_2^2 + (1 + p_2)x_3^2$

where $|p_1| \leq 1$ and $|p_2| \leq 1$. \( \text{(11)} \)

4.1 Controller design and solving parameters

From system (11), it is easy to get

$$\text{Hess}(H_1(x_0)) = \text{Diag}\{2, 2, 2\} > 0, \quad \text{Hess}(H_2(x_0)) = \text{Diag}\{1, 4, 2\} > 0$$

So Assumption 1 holds.

Then, we check that condition (4) holds for all $x$ and given $\gamma$. From system (11), we have

Let $\gamma_1^* = 1$, $\gamma_2^* = \sqrt{2}$. To ensure that condition (4) holds, the following statement should be satisfied.

$$\gamma \geq \max \{\gamma_1^*, \gamma_2^*\} \quad \text{(12)}$$

Next, we consider condition (5) such that system (11) satisfies robustness in $H_\infty$ control. Set the Lyapunov function $V_1(x) = H_1(x)$, $V_2(x) = H_2(x)$. It follows from controller (6) that
where $K_1 = [K_{11} K_{12}]^T$, $K_2 = [K_{21} K_{22}]^T$.

We know $n = 4$ in system (11) and let $r = 1$. We have

$$K_{11} = a_{11}x_1 + a_{12}x_2 + a_{13}x_3, K_{12} = b_{11}x_1 + b_{12}x_2 + b_{13}x_3,$$

$$K_{21} = a_{21}x_1 + a_{22}x_2 + a_{23}x_3, K_{22} = b_{21}x_1 + b_{22}x_2 + b_{23}x_3,$$

where $a_{ij}, b_{ij}, i = 1, 2, j = 1, 2, 3$ are the parameters.

From system (11), we obtain

$$\nabla H_1(x) = [2x_1 - 2x_2 - 2x_3]^T$$

$$\nabla H_2(x) = [x_1 - 4x_2 - 2x_3]^T$$

Let $S_1 = -\nabla H_1^T g_1 K_1$, $S_2 = -\nabla H_2^T g_2 K_2$, we have

$$S_1 = -2a_{11}x_1^2 - (2a_{12} + 2b_{11}) x_1 x_2 - (2a_{13} + 2a_{13}) x_1 x_3 - 2b_{12}x_2^2 - (2a_{12} + 2b_{13}) x_2 x_3 - 2a_{13}x_3^2,$$

$$S_2 = -b_{21}x_1^2 - (4a_{21} + 2b_{21}) x_1 x_2 - (2b_{21} + b_{23}) x_1 x_3 - 4a_{22}x_2^2 - (4a_{22} + 2b_{22}) x_2 x_3 - 2b_{23}x_3^2,$$

where $a_{ij}, b_{ij}, i = 1, 2, j = 1, 2, 3$ are the parameters and can be rewritten as coefficient matrix $M_1$ and $M_2$ (multiply constant 2 for simplifying computation), respectively.

$$M_1 = \begin{bmatrix}
-4a_{11} & -2a_{12} - 2b_{11} & -2a_{13} - 2b_{13} \\
-2a_{12} - 2b_{11} & -4b_{12} & -2a_{12} - 2b_{12} \\
-2a_{13} - 2b_{13} & -2a_{12} - 2b_{13} & -4a_{13} - 2b_{23} \\
-2b_{21} & -4a_{21} - b_{22} & -2b_{21} - b_{22} \\
-4a_{21} - b_{22} & -8a_{22} - 2b_{22} & -4a_{23} - 2b_{23} \\
-2b_{21} - b_{22} & -4a_{23} - 2b_{23} & -4b_{22}
\end{bmatrix}$$

$$M_2 = \begin{bmatrix}
-2a_{11} & -2a_{12} - 2a_{13} & -2a_{13} - 2b_{13} \\
-2a_{12} - 4a_{13} - b_{21} & -2a_{13} - b_{21} & -2b_{21} - b_{23} \\
-2a_{13} - 2b_{13} & -8a_{13} - 2b_{23} & -4a_{23} - 2b_{22} \\
-2b_{21} & -4a_{21} - b_{22} & -2b_{21} - b_{23} \\
-4a_{21} - b_{22} & -8a_{22} - 2b_{22} & -4a_{23} - 2b_{23} \\
-2b_{21} - b_{22} & -4a_{23} - 2b_{23} & -4b_{22}
\end{bmatrix}$$

All principal minors of $M_1$ and $M_2$ must be positive semi-definite, we can obtain a set of parameters solution $U_1$ and $U_2$ using CAD algorithm, respectively.

$$U_1 = \{a_{11} \leq 0, a_{12} = 0, a_{13} = 0, b_{11} = 0, b_{12} \leq 0, b_{13} = 0\}$$

$$U_2 = \{a_{21} = 0, a_{22} \leq 0, a_{23} = 0, b_{21} \leq 0, b_{22} = 0, b_{23} = 2b_{23}\}$$

(15)

(16)

Substitute $U_1$ and $U_2$ into controller (13) and (14), we obtain

$$u_{\lambda(\cdot)} = \begin{bmatrix}
-\frac{1}{2}h_1^2 + \frac{1}{\gamma^2} \left(x_1 + x_2\right) + a_{11}x_1 \\
-\frac{1}{2}h_2^2 + \frac{1}{\gamma^2} \left(x_2 + b_{12} x_2\right)
\end{bmatrix} - \Phi_1 \hat{\phi}_1$$

(17)

$$\hat{\phi} = Q_1 (-2x_1x_3 + 4x_2x_3 - 2x_i^3)$$

(18)

where $a_{11} \leq 0, b_{12} \leq 0$.

So we have a family of controllers of system (11), which have parameters.

4.2 Simulations and results

In order to evaluate the robustness of the controller (17) and (18), we set the parameters of system (11) as: $\gamma = 2, h_1 = 2, h_2 = 3, h_{22} = 2$ and the parameters of controller as: $a_{11} = -10, b_{12} = 10, a_{22} = 10, b_{23} = 10$. Choose $\Phi_1 = [-x_3 2x_3]^T, \Phi_2 = [-x_3, -2x_3]^T, \hat{\phi} = 2p_1, \hat{\phi} = 2p_2$. We have

$$J_1(x, p) - R_1(x, p) \Delta_{h_1}(x, p) = g_1 \Phi_1 \hat{\phi} = -2p_1x_3 + 4p_2x_3 - 2p_1x_3$$

$$J_2(x, p) - R_2(x, p) \Delta_{h_2}(x, p) = g_2 \Phi_2 \hat{\phi} = -4p_2x_3 - 2p_2x_3 - 4p_2x_3$$

So the condition (6) holds for system (11). Let $Q_1 = 1, Q_2 = 1, p_1 = 0.8, p_2 = 0.8$, we obtain the controller (19) and (20).

$$u_{\lambda(\cdot)} = \begin{bmatrix}
\frac{-57}{4} x_1 - \frac{17}{4} x_3 + x_i \hat{\phi}_1 \\
\frac{-77}{4} x_2 - 2x_i \hat{\phi}_1
\end{bmatrix}$$

(19)

$$\hat{\phi}_1 = -2x_1x_3 + 4x_2x_3 - 2x_i^3$$

(20)
Suppose that \( x(0) = (0, 1, 0.4)^T \) is the pre-assigned operating point of system (11), we impose an external disturbance \( \omega = [4.4; 4.4]^T \) on system (11) during the time period 0.6~0.9s and 3~3.6s. The simulation results are shown in Figure. 1 and Figure. 2.

Figure. 1 and Figure. 2 are the response of the state, the controller (19) and (20) and estimate \( \hat{\theta} \) of system (11) under the switching law \( \lambda_i(t) \), respectively.

\[
\lambda_i(t) = \begin{cases} 
2, & t \in [t_{2k}, t_{2k+1}), t_{2k+1} - t_{2k} = 0.15, \\
1, & t \in [t_{2k+1}, t_{2k+2}), t_{2k+2} - t_{2k+1} = 0.15, \\
k = 0, 1, 2, \ldots,
\end{cases}
\]

When the conditions and the parameters of system (11) are not changing, the parameters of controller as: \( a_{11} = -1, b_{12} = -1, a_{22} = -1, b_{21} = -1 \). We obtain the other controller:

\[
u|_{t^{(i)-2}} = \begin{bmatrix}
\frac{57}{2}x_2 + x_3\hat{\theta}_2 \\
\frac{97}{8}x_1 - \frac{97}{4}x_3 + 2x_4\hat{\theta}_2
\end{bmatrix}
\]

\[
\dot{\hat{\theta}} = -2x_1x_3 - 4x_2x_3 - 4x_3^2
\]

\[
\dot{\hat{\theta}} = -2x_1x_3 + 4x_2x_3 - 2x_3^2
\]

\[
\dot{\hat{\theta}} = -2x_1x_3 - 4x_2x_3 - 4x_3^2
\]

Figure. 3 and Figure. 4 are the response of the state, the controller (21) and (22) and estimate \( \hat{\theta} \) of system (11) under the switching law \( \lambda_i(t) \), respectively.

Figure. 1: Swing curves of \( x \) under \( \lambda_i(t) \)

Figure. 2: Swing curves of \( u \) and \( \hat{\theta} \) under \( \lambda_i(t) \)

Figure. 3: Swing curves of \( x \) under \( \lambda_i(t) \)

Figure. 4: Swing curves of \( u \) and \( \hat{\theta} \) under \( \lambda_i(t) \)
From Figure 3 and Figure 1, we know that we can adjust the parameters value in controller (17) and (18) to insure that the system (11) is adaptive $H_{\infty}$ control and optimize the robustness of system (11) under $\lambda_i(t)$.

Then, we consider the robustness of system (11) against external disturbance and parameter perturbations under the arbitrary switching law.

$$\lambda_i(t) = \begin{cases} 1, & t \in [t_{2k}, t_{2k+1}), t_{2k+1} - t_{2k} = 0.4 \times \text{rand}, \\ 2, & t \in [t_{2k+1}, t_{2k+2}), t_{2k+2} - t_{2k+1} = 0.4 \times \text{rand}, \\ k = 0, 1, 2, \ldots, \\ \end{cases}$$

where $0 < \text{rand} < 1$.

Figure 5 and Figure 6 are the response of the state, the controller (21) and (22) and estimate $\hat{\theta}$ of system (11) under the switching law $\lambda_2(t)$, respectively.

5. Conclusions

In this paper, we consider adaptive $H_{\infty}$ control problem for SDHSS under arbitrary switching laws. When there are external disturbances and parameter perturbations in such systems, a family of controllers has been obtained using the MLFs method for such systems and an algorithm for solving parameters of the controller has been proposed with symbolic computation. The numerical experiment and simulations show that the controllers have efficient in adaptive $H_{\infty}$ control.

The method proposed in this paper suggests a framework to solve adaptive $H_{\infty}$ control problem of general nonlinear switched systems.

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