Global Finite-Time Stabilization of Stochastic Nonlinear Systems with Low-Order Nonlinearities

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Abstract: This paper investigates the problem of global finite-time stabilization by state feedback for a class of stochastic nonlinear systems with low-order nonlinearities, to which the existing control methods are inapplicable. By skillfully adopting the method of adding a power integrator and constructing twice continuous differential Lyapunov functions, a stepwise constructive continuous state feedback control methodology is proposed. Based on stochastic finite-time stability theorem, it is proved that the solution of the closed-loop system is finite-time stable in probability. Two simulation examples are provided to illustrate the effectiveness of the proposed approach.

Key–Words: Stochastic nonlinear systems, Low-order nonlinearities, Finite-time stable in probability

1 Introduction

In this paper, we consider the following stochastic nonlinear system:

$$\begin{aligned} dx_i &= h_i x_{i+1}dt + f_i(x,u)dt + g_i^T(x,u)d\omega, \\ &\quad i = 1,\cdots,n-1, \\ dx_n &= h_n u dt + f_n(x,u)dt + g_n^T(x,u)d\omega, \end{aligned} \tag{1}$$

where $x = (x_1,\cdots,x_n)^T \in R^n$, $u \in R$ are the system state and input, respectively. h_i, $i = 1,\cdots,n$ are disturbed virtual control coefficients. ω is an m-dimensional independent standard Wiener process defined on a complete probability space $(\Omega,\mathcal{F},P)$ with Ω being a sample space, $\mathcal{F}$ being a σ-field, and P being a probability measure. The functions $f_i : R^n \times R \to R$ and $g_i : R^n \times R \to R^m$ are continuous and vanish at the origin.

In the past decades, the stability analysis and control design for stochastic nonlinear systems have received a great deal of attention since stochastic modeling has come to play an important role in many branches of science and engineering applications. In [1], some fundamental criteria of stochastic stability have been presented for stochastic nonlinear systems via Lyapunov's direct method. Florchinger in [2] extended Sontag's formula to control stochastic differential systems driven by a Wiener process. The work [3] has developed a methodology for recursive construction of controllers under a quadratic Lyapunov function and a risk-sensitive cost function criterion, while the work [4] has designed a backstepping control law by introducing quartic Lyapunov functions. Since then, the stabilization problem of stochastic nonlinear systems have experienced a breakthrough and a series of research results have been achieved, for example, one can see [5-18] and the references therein. However, it should be mentioned that most of the existing works only consider the feedback stabilizer that makes the trajectories of the systems converge asymptotically to the equilibrium almost surely as the time goes to infinity.

Compared to the asymptotic stabilization, the finite-time stabilization, which renders the trajectories of the closed-loop systems convergent to the origin almost surely in a finite time, has many advantages such as fast response, high tracking precision, and disturbance-rejection properties. Hence it is more meaningful to investigate the finite-time stabilization problem than the classical asymptotical stabilization. Recently, the work [19] has presented the concept of finite-time stability in probability for stochastic systems and proved the stochastic Lyapunov theorem on finite-time stability. Subsequently, the works [20-22] designed continuous state-feedback controllers to guarantee the global finite-time stability in probability for stochastic nonlinear systems with different structures, their assumptions on the system growth can be summarized as the form:

$$|f_i| \le \varphi_i \sum_{j=1}^{i} |x_j|, \quad |g_i| \le \phi_i \sum_{j=1}^{i} |x_j|, \tag{2}$$

where φ_i and ϕ_i are nonnegative smooth functions.

However, from both practical and theoretical points of view, it is somewhat restrictive to require system (1) to satisfy such restriction. To illustrate the limitation, let us consider the following simple system

$$dx_1 = udt + x_1^{4/5} d\omega. \tag{3}$$

Due to the presence of low-order term $x_1^{4/5}$, it is easily verified that there are not smooth functions φ_1 and ϕ_1 such that the condition (2) holds. This means that the works [20- 22] cannot lead to any finite-time stabilizer for the system (3). Immediately, the following interesting questions are proposed: *Is it possible to relax the nonlinear growth condition (2) to cover the low-order nonlinearities? Under the weaker condition, how can one design a continuous finite-time stabilizer for the nonlinear system (3) and more general nonlinear system (1) ?*

Motivated by the continuous control ideas in [23,24], and by necessarily modifying the method of adding a power integrator, we shall solve the above problems here. The main contributions of this paper are two-folds: (i) By comparison with the existing results in [20-23], the nonlinear growth condition is largely relaxed and a much weaker sufficient condition is given. (ii) By successfully overcoming some essential difficulties such as the weaker assumption on the system growth and the construction of a C^2, positive-definite and proper Lyapunov function, a new method to global finite-time stabilization of stochastic nonlinear systems by state feedback is given, which can not only be seen as a natural unification of the existing methods, but also leads to more general results never achieved before.

Notations. Throughout this paper, the following notations are adopted. R^+ denotes the set of all nonnegative real numbers, R^n denotes the real n-dimensional stands for the set of all real numbers Euclidean space and $R^{n\times m}$ denotes the space of real $n \times m$-matrixes. R^+_{odd} :=$\{\frac{p}{q} \mid p$ and q positive integers $\}$, R^+_{even}:=$\{\frac{p}{q} \mid p$ is a even positive integer and q is an positive odd integer $\}$, $R^{\geq 2}_{odd}$:=$\{\frac{p}{q} \mid p$ and q are positive odd integers, and $p \geq 2q\}$. For a given vector or matrix X, X^T denotes its transpose, $Tr\{X\}$ denotes its trace when X is square, and $|X|$ is the Euclidean norm of a vector X. C^i denotes the set of all functions with continuous ith partial derivatives; $C^2(R^n)$ denotes the family of all nonnegative functions $V(x)$ on R^n which are C^2 in x; C^2 denotes the family of all functions which are C^2 in the argument. $\mathcal{K}$ denotes the set of all functions: $R^+ \to R^+$, which are continuous, strictly increasing and vanishing at zero; $\mathcal{K}_\infty$ denotes the set of all functions which are of class $\mathcal{K}$ and unbounded. Besides, let $\bar{x}_i = (x_1, \cdots, x_i)^T$ and the arguments of the functions will be omitted or simplified, whenever no confusion can arise from the context. For instance, we sometimes denote a function $f(x(t))$ by simply $f(x)$, $f(\cdot)$ or f.

2 Preliminary results

Consider the stochastic nonlinear system

$$dx = f(x)dt + g(x)d\omega, \tag{4}$$

where $x \in R^n$ is the system state with the initial condition $x(0) = x_0$; ω is an m-dimensional independent standard Wiener process defined on a complete probability space $(\Omega, \mathcal{F}, P)$ with Ω being a sample space, $\mathcal{F}$ being a σ-field, and P being a probability measure. The functions:$f : R^n \to R^n$ and $g : R^n \to R^{n\times m}$ are continuous in x satisfying $f(0) = 0$ and $g(0) = 0$.

For any given $V(x) \in C^2$, associated with stochastic system (4), the the second-order differential operator $\mathcal{L}$ is defined as follows:

$$\mathcal{L}V = \frac{\partial V}{\partial x} f + \frac{1}{2} Tr\Big\{ g^T \frac{\partial^2 V}{\partial x^2} g \Big\}.$$

Definition 1[21]**.** The trivial solution of (4) is said to be finite-time stable in probability if the stochastic system admits a solution for any initial data $x_0 \in R^n$, denoted by $x(t, x_0)$, and the following statements hold:

(i) Finite-time attractive in probability: For every initial value $x_0 \in R^n \setminus \{0\}$, the first hitting time $\tau_{x_0} = \inf\{t : x(t, x_0) = 0\} = \inf\{t : |x(t, x_0)| = 0\}$ called stochastic settling time, is finite almost surely, that is, $P\{= \inf\{t : x(t, x_0) = 0\} < \infty\} = 1$.

(ii) Stable in probability: For every pair of $\varepsilon \in (0, 1)$ and $r > 0$, there exists $\delta = \delta(\varepsilon, r) > 0$ such that $P\{|x(t, x_0)| < r, \forall t \geq 0\} \geq 1 - \varepsilon$,whenever $|x_0| < \delta$.

Lemma 1 [21]**.** Suppose that there exists a nonnegative function $V \in C^2(R^n)$, which is radially unbounded, that is, $\lim_{|x|\to\infty} V(x) = +\infty$. If the second-order differential operator of V with respect to (4) satisfies $\mathcal{L}V \leq 0$, then (4) has a solution for any initial data.

Lemma 2[21]**.** Assume that (4) admits a solution for each initial vector. If there exists a C^2 function $V : R^n \to R^+$, $\mathcal{K}_\infty$ class functions μ_1 and μ_2, real numbers $c > 0$ and $0 < \alpha < 1$, such that for all $t > 0$,

$$\mu_1(|x|) \leq V(x) \leq \mu_2(|x|),$$

$$\mathcal{L}V(x) \leq -cV^\alpha(x),$$

then the origin of system (4) is globally finite-time stable in probability.

In the remainder of this section, we present the following lemmas which play an important role in the design process.

Lemma 3 [25]**.** For $x \in R$, $y \in R$, and $p \geq 1$ being a constant, the following inequalities hold:

$$|x+y|^p \leq 2^{p-1}|x^p + y^p|,$$

$$(|x|+|y|)^{1/p} \leq |x|^{1/p}+|y|^{1/p} \leq 2^{(p-1)/p}(|x|+|y|)^{1/p}.$$

If $p \geq 1$ is odd, then

$$|x-y|^p \leq 2^{p-1}|x^p - y^p|,$$

$$|x^{1/p} - y^{1/p}| \leq 2^{(p-1)/p}(|x-y|)^{1/p}.$$

Lemma 4 [26]**.** Let x, y be real variables, then for any positive real numbers a, m and n, one has

$$a|x|^m|y|^n \leq b|x|^{m+n} + \frac{n}{m+n}\Big(\frac{m+n}{m}\Big)^{-\frac{m}{n}} a^{\frac{m+n}{n}} b^{-\frac{m}{n}} |y|^{m+n},$$

where $b > 0$ is any real number.

3 Control design and stability analysis

3.1 Assumptions

The objective of this paper is to develop a recursive design method for globally finite-time stabilizing system (1) via continuous state feedback under the following assumptions.

Assumption 1. The signs of the constants h_i, $i = 1, \cdots, n$ are known, and there exist two known positive constants h_{i1} and h_{i2} such that

$$h_{i1} \leq |h_i| \leq h_{i2}.$$

Assumption 2. For $i = 1, \cdots, n$, there are smooth functions $\varphi_i(\bar{x}_i) \geq 0$, $\phi_i(\bar{x}_i) \geq 0$ and a constant $\tau \in (-\frac{1}{n}, 0)$ such that

$$\begin{aligned} |f_i(x,u)| &\leq \varphi_i(\bar{x}_i)\sum_{j=1}^{i}|x_j|^{\frac{r_i+\tau}{r_j}}, \\ |g_i(x,u)| &\leq \phi_i(\bar{x}_i)\sum_{j=1}^{i}|x_j|^{\frac{2r_i+\tau}{2r_j}}, \end{aligned} \tag{5}$$

where $r_1 = 1$ and $r_{i+1} = r_i + \tau > 0$, $i = 1, \cdots, n$.

For simplicity, it is assumed that $\tau = -\frac{m}{n}$ with m being any even integer and m being any odd integer, under which and the definition of r_i in Assumption 2, we know that $r_i \in R^+_{odd}$.

Remark 1. It is worth pointing out that Assumption 2, which gives the nonlinear growth condition on the system drift and the diffusion terms, encompasses the assumptions in the closely related works[20-23]. To clearly show this, we would like to make the following comparisons to reveal the relationship between Assumption 2 and the counterparts in [20-23], that is, Assumption 2 includes those as special cases:

(i) In [20-22], the system nonlinearities f_i's and g_i's are required to satisfy:

$$\begin{aligned} |f_i(x,u)| &\leq \gamma_i(\bar{x}_i)(|x_1| + \cdots + |x_i|), \\ |g_i(x,u)| &\leq \eta_i(\bar{x}_i)(|x_1| + \cdots + |x_i|), \end{aligned} \tag{6}$$

where $\gamma_i(\bar{x}_i) \geq 0$ and $\eta_i(\bar{x}_i) \geq 0$, $i = 1, \cdots, n$ are C^2 functions. From the definitions of r_i's and τ, we get $0 < \frac{r_i+\tau}{r_j} = \frac{1+i\tau}{1+(j-1)\tau} < 1$ and $0 < \frac{2r_i+\tau}{2r_j} = \frac{2+(2i-1\tau}{2+2(j-1)\tau} < 1$ which mean that

$$\begin{aligned} |f_i(x,u)| &\leq \gamma_i(\bar{x}_i)(|x_1| + \cdots + |x_i|) \\ &\leq \gamma_i(\bar{x}_i)(|x_1|^{\frac{r_i+\tau}{r_1}}|x_1|^{1-\frac{r_i+\tau}{r_1}} + \cdots \\ &\quad + |x_i|^{\frac{r_i+\tau}{r_i}}|x_i|^{1-\frac{r_i+\tau}{r_i}}) \\ &\leq \varphi_i(\bar{x}_i)(|x_1|^{\frac{r_i+\tau}{r_1}} + \cdots + |x_i|^{\frac{r_i+\tau}{r_i}}), \end{aligned}$$

$$\begin{aligned} |g_i(x,u)| &\leq \eta_i(\bar{x}_i)(|x_1| + \cdots + |x_i|) \\ &\leq \eta_i(\bar{x}_i)(|x_1|^{\frac{2r_i+\tau}{2r_1}}|x_1|^{1-\frac{2r_i+\tau}{2r_1}} + \cdots \\ &\quad + |x_i|^{\frac{2r_i+\tau}{2r_i}}|x_i|^{1-\frac{2r_i+\tau}{2r_i}}) \\ &\leq \phi_i(\bar{x}_i)(|x_1|^{\frac{2r_i+\tau}{2r_1}} + \cdots + |x_i|^{\frac{2r_i+\tau}{2r_i}}), \end{aligned}$$

where $\varphi_i(\bar{x}_i) \geq \max\{\gamma_i|x_1|^{1-\frac{r_i+\tau}{r_1}}, \cdots, \gamma_i|x_i|^{1-\frac{r_i+\tau}{r_i}}\}$ and $\phi_i(\bar{x}_i) \geq \max\{\eta_i|x_1|^{1-\frac{2r_i+\tau}{2r_1}}, \cdots, \eta_i|x_1|^{1-\frac{2r_i+\tau}{2r_i}}\}$ are smooth functions, hence the main assumption (6) in [20-22] is a special case of Assumption 2 above.

(ii) In [23], the system nonlinearities f_i's and g_i's are required to satisfy:

$$\begin{aligned} |f_i(x,u)| &\leq a_1\sum_{j=1}^{i}|x_j|^{\frac{r_i+\tau}{r_j}}, \\ |g_i(x,u)| &\leq a_2\sum_{j=1}^{i}|x_j|^{\frac{2r_i+\tau}{2r_j}}, \end{aligned} \tag{7}$$

with constants a_1, $a_2 > 0$ and $\tau \in (-\frac{1}{n}, 0)$. Obviously, when $\varphi_i(\bar{x}_i) = a_1$ and $\eta_i(\bar{x}_i) = a_2$, inequality (5) degenerates to inequality (7). Thus, the main assumption (7) in [23] is a special case of Assumption 2 above.

Remark 2. Assumption 1 slightly relaxes the control coefficients in [20-23], where all of them are precisely 1, but makes the finite-time control design of system (1) more complicated. Moreover, Assumption 1 implies that the signs of h_i's are known and remain unchanged. Thus, without loss of generality, we suppose all h_i's are positive, that is, $h_i > 0$, $i = 0, \cdots, n$.

3.2 Control design procedure

In this subsection, we shall construct a continuous state feedback controller by using the method of adding a power integrator.

Step 1. Let $\xi_1 = x_1^{\frac{\sigma}{r_1}}$ with $\sigma \in R_{odd}^{\geq 2}$ and choose Lyapunov function

$$V_1 = W_1 = \int_{x_1^*}^{x_1} \left(s^{\frac{\sigma}{r_1}} - x_1^{*\frac{\sigma}{r_1}}\right)^{\frac{4l\sigma-\tau-r_1}{\sigma}} ds, \quad (8)$$

where $x_1^* = 0$ and l is a constant satisfying $4l \in R_{even}^+$ and $(4l-2)\sigma \geq 1$. Using (1) and (5), we have

$$\mathcal{L}V_1 \leq h_1 x_1^{\frac{4l\sigma-\tau-r_1}{r_1}}(x_2 - x_2^*) + h_1 x_1^{\frac{4l\sigma-\tau-r_1}{r_1}} x_2^* + x_1^{4l\sigma}\Big(\varphi_1 + \frac{4l\sigma-\tau-r_1}{2r_1}\phi_1^2\Big). \quad (9)$$

Obviously, the C^0 virtual controller

$$x_2^* = -\frac{1}{h_{11}}\Big(\lambda_1 + \varphi_1 + \frac{4l\sigma-\tau-r_1}{2r_1}\phi_1^2\Big)x_1^{r_1+\tau} := -\beta_1^{\frac{r_2}{\sigma}}\xi_1^{\frac{r_2}{\sigma}} \quad (10)$$

where λ_1 is a positive design constant, results in

$$\mathcal{L}V_1 \leq -\lambda_1\xi_1^{4l} + h_1\xi_1^{\frac{4l\sigma-\tau-r_1}{\sigma}}(x_2 - x_2^*). \quad (11)$$

Inductive step. Suppose at step $k-1$, there are a C^2, proper and positive definite Lyapunov function V_{k-1}, and a set of virtual controllers $x_1^*, \cdots, x_k^*$ defined by

$$\begin{array}{lllllll} x_1^* &=& 0, & \xi_1 &=& x_1^{\frac{\sigma}{r_1}} - x_1^{*\frac{\sigma}{r_1}}, \\ x_2^* &=& -\beta_1^{\frac{r_2}{\sigma}}\xi_1^{\frac{r_2}{\sigma}}, & \xi_2 &=& x_2^{\frac{\sigma}{r_2}} - x_2^{*\frac{\sigma}{r_2}}, \\ \vdots & & & \vdots & & \\ x_k^* &=& -\beta_{k-1}^{\frac{r_k}{\sigma}}\xi_{k-1}^{\frac{r_k}{\sigma}}, & \xi_k &=& x_k^{\frac{\sigma}{r_k}} - x_k^{*\frac{\sigma}{r_k}}, \end{array} \quad (12)$$

with $\beta_1 > 0, \cdots, \beta_{k-1} > 0$ being smooth, such that

$$\mathcal{L}V_{k-1} \leq -\sum_{i=1}^{k-1}\Big(\lambda_i - \sum_{m=i+1}^{k-1} l_m\Big)\xi_i^{4l} + h_{k-1}\xi_{k-1}^{\frac{4l\sigma-\tau-r_{k-1}}{\sigma}}(x_k - x_k^*). \quad (13)$$

whereλ_i, $i = 1, \cdots, k-1$, l_m, $m = 2, \cdots, k-1$ are positive design constants and $\sum_{m=i+1}^{k} l_m = 0$ for the case of $k = 2$.

To complete the induction, at the kth step, we choose the following Lyapunov function

$$V_k(\bar{x}_k) = V_{k-1}(\bar{x}_{k-1}) + W_k(\bar{x}_k), \quad (14)$$

where

$$W_k(\bar{x}_k) = \int_{x_k^*}^{x_k} \left(s^{\frac{\sigma}{r_k}} - x_k^{*\frac{\sigma}{r_k}}\right)^{\frac{4l\sigma-\tau-r_k}{\sigma}} ds. \quad (15)$$

Note that

$$x_k^{*\frac{\sigma}{r_k}} = -\beta_{k-1}\xi_{k-1} = -\sum_{l=1}^{k-1} B_l x_l^{\frac{\sigma}{r_l}}, \quad (16)$$

where $B_l = \beta_{k-1}\cdots\beta_l, l = 1, \cdots, k-1$ are smooth functions, and $\sigma/r_l > 2$, using a similar method to the one in [15], the function V_k can be shown to be C^2, proper and positive definite. Moreover, we can obtain

$$\begin{aligned}
\frac{\partial W_k}{\partial x_k} &= \xi_k^{\frac{4l\sigma-\tau-r_k}{\sigma}}, \\
\frac{\partial^2 W_k}{\partial x_k^2} &= \frac{4l\sigma-\tau-r_k}{r_k}\xi_k^{\frac{(4l-1)\sigma-\tau-r_k}{\sigma}} x_k^{\frac{\sigma-r_k}{r_k}}, \\
\frac{\partial^2 W_k}{\partial x_k \partial x_i} &= \frac{\partial^2 W_k}{\partial x_i \partial x_k} = -\frac{4l\sigma-\tau-r_k}{\sigma} \\
&\quad \times \xi_k^{\frac{(4l-1)\sigma-\tau-r_k}{\sigma}} \frac{\partial(x_k^{*\frac{\sigma}{r_k}})}{\partial x_i}, \\
\frac{\partial W_k}{\partial x_i} &= -\frac{4l\sigma-\tau-r_k}{\sigma}\frac{\partial(x_k^{*\frac{\sigma}{r_k}})}{\partial x_i} \\
&\quad \times \int_{x_k^*}^{x_k}\left(s^{\frac{\sigma}{r_k}} - x_k^{*\frac{\sigma}{r_k}}\right)^{\frac{(4l-1)\sigma-\tau-r_k}{\sigma}} ds, \\
\frac{\partial^2 W_k}{\partial x_i^2} &= \frac{4l\sigma-\tau-r_k}{\sigma}\cdot\frac{(4l-1)\sigma-\tau-r_k}{\sigma} \\
&\quad \times \Big(\frac{\partial(x_k^{*\frac{\sigma}{r_k}})}{\partial x_i}\Big)^2 \\
&\quad \times \int_{x_k^*}^{x_k}\left(s^{\frac{\sigma}{r_k}} - x_k^{*\frac{\sigma}{r_k}}\right)^{\frac{(4l-2)\sigma-\tau-r_k}{\sigma}} ds \\
&\quad -\frac{4l\sigma-\tau-r_k}{\sigma}\frac{\partial^2(x_k^{*\frac{\sigma}{r_k}})}{\partial x_i^2} \\
&\quad \times \int_{x_k^*}^{x_k}\left(s^{\frac{\sigma}{r_k}} - x_k^{*\frac{\sigma}{r_k}}\right)^{\frac{(4l-1)\sigma-\tau-r_k}{\sigma}} ds, \\
\frac{\partial^2 W_k}{\partial x_i \partial x_j} &= \frac{4l\sigma-\tau-r_k}{\sigma}\cdot\frac{(4l-1)\sigma-\tau-r_k}{\sigma} \\
&\quad \times \frac{\partial(x_k^{*\frac{\sigma}{r_k}})}{\partial x_i}\frac{\partial(x_k^{*\frac{\sigma}{r_k}})}{\partial x_j} \\
&\quad \times \int_{x_k^*}^{x_k}\left(s^{\frac{\sigma}{r_k}} - x_k^{*\frac{\sigma}{r_k}}\right)^{\frac{(4l-2)\sigma-\tau-r_k}{\sigma}} ds,
\end{aligned} \quad (17)$$

for $i, j = 1, \cdots, k-1$, $i \neq j$.

Using (13), (14) and (15), it follows that

$$\begin{aligned}\mathcal{L}V_k \le & -\sum_{i=1}^{k-1}\Big(\lambda_i - \sum_{m=i+1}^{k-1} l_m\Big)\xi_i^{4l} \\ & + h_k \xi_k^{\frac{4l\sigma-\tau-r_k}{\sigma}} x_{k+1} \\ & + h_{k-1}\xi_{k-1}^{\frac{4l\sigma-\tau-r_{k-1}}{\sigma}}(x_k - x_k^*) \\ & + \xi_k^{\frac{4l\sigma-\tau-r_k}{\sigma}} f_k + \sum_{i=1}^{k-1}\frac{\partial W_k}{\partial x_i}(h_i x_{i+1} + f_i) \\ & + \frac{1}{2}\sum_{i,j=1,i\neq j}^{k-1}\Big|\frac{\partial^2 W_k}{\partial x_i \partial x_j}\Big||g_i^T g_j| \\ & + \frac{1}{2}\sum_{i=1}^{k-1}\Big|\frac{\partial^2 W_k}{\partial x_i^2}\Big||g_i|^2 \\ & + \frac{1}{2}\sum_{i=1}^{k-1}\Big|\frac{\partial^2 W_k}{\partial x_k \partial x_i}\Big||g_k^T g_i| + \frac{1}{2}\Big|\frac{\partial^2 W_k}{\partial x_k^2}\Big||g_k|^2.\end{aligned} \tag{18}$$

In order to proceed further, an appropriate estimate shouldbe given for the last seven terms on the right-hand side of (18). This is accomplished in the following propositions whose technical proof are given in Appendix.

Proposition 1. There exist a constant $l_k > 0$ and a smooth function $\varphi_k \ge 0$ such that

$$\begin{aligned} & h_{k-1}\xi_{k-1}^{\frac{4l\sigma-\tau-r_{k-1}}{\sigma}}(x_k - x_k^*) + \xi_k^{\frac{4l\sigma-\tau-r_k}{\sigma}} f_k \\ & + \sum_{i=1}^{k-1}\frac{\partial W_k}{\partial x_i}(h_i x_{i+1} + f_i) \\ & + \frac{1}{2}\sum_{i,j=1,i\neq j}^{k-1}\Big|\frac{\partial^2 W_k}{\partial x_i \partial x_j}\Big||g_i^T g_j| + \frac{1}{2}\sum_{i=1}^{k-1}\Big|\frac{\partial^2 W_k}{\partial x_i^2}\Big||g_i|^2 \\ & + \frac{1}{2}\sum_{i=1}^{k-1}\Big|\frac{\partial^2 W_k}{\partial x_k \partial x_i}\Big||g_k^T g_i| + \frac{1}{2}\Big|\frac{\partial^2 W_k}{\partial x_k^2}\Big||g_k|^2 \\ \le & \; l_k \sum_{i=1}^{k-1}\xi_i^{4l} + \xi_k^{4l}\varphi_k.\end{aligned} \tag{19}$$

Substituting (19) into (18) yields

$$\begin{aligned}\mathcal{L}V_k \le & -\sum_{i=1}^{k-1}\Big(\lambda_i - \sum_{m=i+1}^{k} l_m\Big)\xi_i^{4l} \\ & + h_k \xi_k^{\frac{4l\sigma-\tau-r_k}{\sigma}}(x_{k+1} - x_{k+1}^*) \\ & + h_k \xi_k^{\frac{4l\sigma-\tau-r_k}{\sigma}} x_{k+1}^* + \xi_k^{4l}\varphi_k.\end{aligned} \tag{20}$$

Clearly, the C^0 virtual controller

$$\begin{aligned} x_{k+1}^* & = -\frac{1}{h_{k1}}(\lambda_k + \varphi_k)\xi_k^{\frac{r_k+\tau}{\sigma}} \\ & := -\beta_k^{\frac{r_{k+1}}{\sigma}}\xi_k^{\frac{r_{k+1}}{\sigma}},\end{aligned} \tag{21}$$

where λ_k is a positive design constant, results in

$$\begin{aligned}\mathcal{L}V_k \le & -\sum_{i=1}^{k}\Big(\lambda_i - \sum_{m=i+1}^{k} l_m\Big)\xi_i^{4l} \\ & + h_k \xi_k^{\frac{4l\sigma-\tau-r_k}{\sigma}}(x_{k+1} - x_{k+1}^*).\end{aligned} \tag{22}$$

This completes the proof of the inductive step.

Using the inductive argument above, we conclude that at the nth step, there exist a continuous state feedback control law of the form

$$u = x_{n+1}^* = -\beta_n^{\frac{r_{n+1}}{\sigma}}\xi_n^{\frac{r_{n+1}}{\sigma}}, \tag{23}$$

with $\beta_n > 0$ being smooth, such that

$$\mathcal{L}V_n \le -\sum_{i=1}^{n}\Big(\lambda_i - \sum_{m=i+1}^{n} l_m\Big)\xi_i^{4l}. \tag{24}$$

It is clear that by choosing λ_i's as

$$\lambda_i > \sum_{m=i+1}^{n} l_m, \tag{25}$$

we lead to

$$\mathcal{L}V_n \le -\lambda_0 \sum_{i=1}^{n}\xi_i^{4l}. \tag{26}$$

where $\lambda_0 = \min\{\lambda_i - \sum_{m=i+1}^{n} l_m\} > 0$.

We have thus far completed the controller design procedure. The results can be summarized into the following theorem.

Theorem 1. Under Assumptions 1 and 2, there exists a continuous state feedback controller (23) such that the origin of system (1) is globally finite-time stable in probability.

Proof. By using Lemma 3, it is easy to see that

$$\begin{aligned} W_k & = \int_{x_k^*}^{x_k}\Big(s^{\frac{\sigma}{r_k}} - x_k^{*\frac{\sigma}{r_k}}\Big)^{\frac{4l\sigma-\tau-r_k}{\sigma}} ds \\ & \le |\xi_k|^{\frac{4l\sigma-\tau-r_k}{\sigma}}|x_k - x_k^*| \\ & \le 2^{1-\frac{r_k}{\sigma}}|\xi_k|^{\frac{4l\sigma-\tau}{\sigma}}.\end{aligned} \tag{27}$$

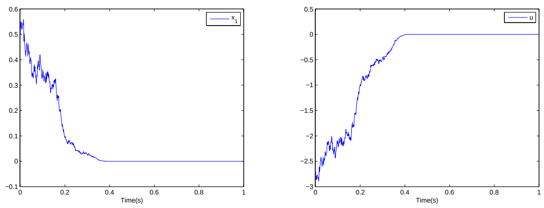
So we have the following estimate

$$V_n = \sum_{k=1}^{n} W_k \le 2\sum_{k=1}^{n}|\xi_k|^{\frac{4l\sigma-\tau}{\sigma}}. \tag{28}$$

Let $\alpha = 4l\sigma/(4l\sigma - \tau)$. With (28) and (26) in mind, by Lemma 2, it is not difficult to obtain that

$$\mathcal{L}V_n \le -\lambda_0 V_n^{\alpha}/2^{\alpha}. \tag{29}$$

Thus, according to Lemmas 1 and 2, under the continuous state feedback controller (23), the origin of system (1) is finite-time stable in probability.



(a) System states (b) Control input

Figure 1: The responses of the closed-loop system (30) and (31).

4 Simulation examples

In this section, two examples are given to illustrate of the effectiveness of the proposed approach in section 3.

Example 1. Consider the following one-dimensional system

$$dx_1 = udt + x_1^{\frac{4}{5}} d\omega. \tag{30}$$

As shown in the Introduction, the works [20-22] cannot provide us a global finite-time stabilizing controller. However, it is easily verified that $|g_1| \le |x_1|^{\frac{4}{5}}$, therefore Assumption 2 holds with $\varphi_1 = 0$, $\phi_1 = 1$ and $\tau = -\frac{2}{5}$. By choosing $\sigma = \frac{11}{5}$ and $l = 1$, we apply the design procedure shown in Section 3 to system (30) and obtain the following continuous controller

$$u = -4.1 x_1^{\frac{3}{5}}. \tag{31}$$

For the simulation, we choose the initial values $x_1(0) = 0.5$. Figure 1 gives the responses of (30) and (31), from which the efficiency of the controller is demonstrated.

Example 2. Consider the parallel active suspension system with random noise[20], which is described by

$$\begin{aligned} dx_1 &= \frac{1}{A} x_2 dt + g_1 d\omega, \\ dx_2 &= k_f i_v dt - c_f x_2 dt + g_2 d\omega, \end{aligned} \tag{32}$$

where x_1 is the suspension travel, x_2 is the fluid flow into the hydraulic actuator, A is the effective surface of piston, c_f and k_f are some positive constants, and i_v is the current input that adjusts the opening of the current-controlled solenoid valve that controls the fluid flow. Obviously, system (32) is in the form of system (1) with $h_1 = \frac{1}{A}$, $h_2 = k_f$, $f_1 = 0$ and $f_2 = -c_f x_2$. To illustrate our design scheme, we choose the following parameters: $A = k_f = c_f = 1$, $g_1 = \frac{1}{5} x_1^{\frac{6}{7}}$ and $g_2 = \frac{1}{2} x_1^2$. Now, the dynamics of the suspension system with random noise (32) can be rewritten as

$$\begin{aligned} dx_1 &= x_2 dt + \frac{1}{5} x_1^{\frac{6}{7}} d\omega, \\ dx_2 &= i_v dt - x_2 dt + \frac{1}{2} x_1^2 d\omega, \end{aligned} \tag{33}$$

It is worth pointing out that although system (33) is simple, it cannot be globally finite-time stabilized using the design methods presented in [20-23] because of the presence of both low-order term $\frac{1}{5} x_1^{\frac{6}{7}}$ and high-order term $\frac{1}{2} x_1^2$. Choose $\tau = -\frac{2}{7}$, which together with $r_1 = 1$ implies that $r_2 = \frac{5}{7}$ and $r_3 = \frac{3}{7}$. By Lemma 4, it is easy to get $|g_1| \le \frac{1}{5} |x_1|^{\frac{6}{7}}$, $|f_2| \le (1 + x_2^2)|x_2|^{\frac{3}{5}}$ and $|g_2| \le \frac{1}{2}(1 + x_1^2)|x_1|^{\frac{4}{7}}$. Clearly, Assumption 1 and 2 hold with $h_{11} = h_{12} = h_{21} = h_{22} = 1$, $\varphi_1 = 0$, $\phi_1 = \frac{1}{5}$, $\varphi_2 = 1 + x_2^2$ and $\phi_2 = \frac{1}{2}(1 + x_1^2)$. In the following design procedure, we choose $l = \frac{2}{3}$ and $\sigma = \frac{15}{7}$.

Defining $\xi_1 = x_1^{\frac{15}{7}}$ and choosing $V_1 = W_1 = \int_{x_1^*}^{x_1} (s^{\frac{15}{7}} - x_1^{*\frac{15}{7}})^{\frac{7}{3}} ds$ with $x_1^* = 0$, we obtain

$$\mathcal{L}V_1 \le -\frac{11}{10} \xi_1^{\frac{8}{3}} + \xi_1^{\frac{7}{3}} (x_2 - x_2^*), \tag{34}$$

where $x_2^* = -\frac{12}{10} x_1^{\frac{5}{7}} := -\beta_1^{\frac{1}{3}} \xi_1^{\frac{1}{3}}$. Defining $\xi_2 = x_2^3 - x_2^{*3}$ and choosing $V_2 = V_1 + W_2 = V_1 + \int_{x_2^*}^{x_2} (s^3 - x_2^{*3})^{\frac{37}{15}} ds$, a direct calculation leads to

$$\begin{aligned} \mathcal{L}V_2 \le & -\frac{11}{10} \xi_1^{\frac{8}{3}} + \xi_1^{\frac{7}{3}} (x_2 - x_2^*) \\ & + \xi_2^{\frac{37}{15}} i_v + \xi_2^{\frac{37}{15}} f_2 + \frac{\partial W_2}{\partial x_1} x_2 \\ & + \frac{1}{2} \frac{\partial^2 W_2}{\partial x_2^2} |g_2|^2 + \frac{1}{2} \frac{\partial^2 W_2}{\partial x_1^2} |g_1|^2 \\ & + \frac{1}{2} \frac{\partial^2 W_2}{\partial x_1 \partial x_2} |g_2^T g_1|. \end{aligned} \tag{35}$$

By using Lemmas 3 and 4, we get

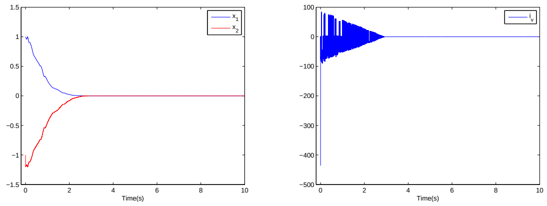
$$\xi_1^{\frac{7}{3}} (x_2 - x_2^*) \le \frac{1}{2} \xi_1^{\frac{8}{3}} + 253.3021 \xi_2^{\frac{8}{3}},$$

$$\xi_2^{\frac{37}{15}} f_2 \le \frac{1}{8} \xi_1^{\frac{8}{3}} + 1.9989(1 + x_2^4) \xi_2^{\frac{8}{3}},$$

$$\frac{1}{2} \frac{\partial^2 W_2}{\partial x_2^2} |g_2|^2 \le \frac{1}{16} \xi_1^{\frac{8}{3}} + 100.7592(1 + x_1^8) \xi_2^{\frac{8}{3}},$$

$$\frac{1}{2} \frac{\partial^2 W_2}{\partial x_1^2} |g_1|^2 \le \frac{1}{16} \xi_1^{\frac{8}{3}} + 4.6914(1 + x_1^4) \xi_2^{\frac{8}{3}},$$

$$\frac{1}{2} \frac{\partial^2 W_2}{\partial x_1 \partial x_2} |g_2^T g_1| \le \frac{1}{16} \xi_1^{\frac{8}{3}} + 1.3304 \xi_2^{\frac{8}{3}},$$



(a) System states (b) Control input

Figure 2: The responses of the closed-loop system (33) and (37).

$$\frac{\partial W_2}{\partial x_1}x_2 \leq \frac{1}{8}\xi_1^{\frac{8}{3}} + 99.6549\xi_2^{\frac{8}{3}},$$

under which

$$\begin{aligned}\mathcal{L}V_2 \leq & -\frac{1}{10}(\xi_1^{\frac{8}{3}} + \xi_2^{\frac{8}{3}}) + \xi_2^{\frac{37}{15}} i_v \\ & +(461.8369 + x_1^4 + x_1^8 + x_2^4)\xi_2^{\frac{8}{3}}.\end{aligned} \tag{36}$$

Therefore, the controller can be chosen as

$$i_v = -(461.8369 + x_1^4 + x_1^8 + x_2^4)\xi_2^{\frac{1}{5}}, \tag{37}$$

which guarantees the origin of the closed-loop system (33) and (37) finite-time stable in probability. With initial values $x_1(0) = 1$ and $x_2(0) = -1$, Figure 2 is obtained to demonstrate the effectiveness of the approach.

5 Conclusion

This paper relaxed the assumptions proposed in [20-23] and obtained much more general results on global finite-time state feedback stabilization of stochastic nonlinear systems than the previous ones. There are some related problems to investigate, e.g., for system (1) with unknown parameters, can an adaptive stabilizing controller be given under a similar assumption? In recent years, many results on deterministic systems have been achieved[27-34], considering that stochastic noise frequently arises and is inevitable in various realistic dynamic models of practical control problems, naturally, how extending these methods to the stochastic counterpart is very interesting and significant.

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Appendix

Proof of Proposition 1. Recalling $\sigma/r_k > 2$ and using (12), it follows from Lemma 3 that

$$\begin{aligned}|x_k - x_k^*| &\leq 2^{1-\frac{r_k}{\sigma}}\left|x_k^{\frac{\sigma}{r_k}} - x_k^{*\frac{\sigma}{r_k}}\right|^{\frac{r_k}{\sigma}} \\ &= 2^{1-\frac{r_k}{\sigma}}|\xi_k|^{\frac{r_k}{\sigma}}.\end{aligned} \tag{A1}$$

By (A1) and Lemma 4, it can be obtained that

$$\begin{aligned}& h_k\xi_{k-1}^{\frac{4l\sigma-\tau-r_{k-1}}{\sigma}}(x_k - x_k^*) \\ & \leq 2^{1-\frac{r_k}{\sigma}}h_{k2}|\xi_{k-1}|^{\frac{4l\sigma-\tau-r_{k-1}}{\sigma}}|\xi_k|^{\frac{r_k}{\sigma}} \\ & \leq \xi_{k-1}^{4l}l_{k1} + \xi_k^{4l}\rho_{k1}.\end{aligned} \tag{A2}$$

where l_{k1} and ρ_{k1} are positive constant.

According to (5), (12) and Lemma 3, it follows that

$$\begin{aligned}|f_i| &\leq \varphi_i\sum_{j=1}^{i}|x_j(t)|^{\frac{r_i+\tau}{r_j}} \\ &\leq \varphi_i\sum_{j=1}^{i}\Big(|\xi_j| + \beta_{j-1}|\xi_{j-1}|\Big)^{\frac{r_i+\tau}{\sigma}} \\ &\leq \bar{\varphi}_i\sum_{j=1}^{i}|\xi_j|^{\frac{r_i+\tau}{\sigma}},\end{aligned} \tag{A3}$$

where $\beta_0 = \xi_0 = 0$ and $\bar{\varphi}_i = \sum_{j=1}^{i}(1 + \beta_{j-1}^{\frac{r_i+\tau}{\sigma}})\varphi_i \geq 0$ is a smooth function.

Using (A3) and Lemma 4, we lead to

$$\begin{aligned}\xi_k^{\frac{4l\sigma-\tau-r_k}{\sigma}}f_k &\leq \bar{\varphi}_k\sum_{j=1}^{k}|\xi_k|^{\frac{4l\sigma-\tau-r_k}{\sigma}}|\xi_j|^{\frac{r_k+\tau}{\sigma}} \\ &\leq l_{k2}\sum_{i=1}^{k-1}\xi_i^{4l} + \xi_k^{4l}\rho_{k2}.\end{aligned} \tag{A4}$$

where l_{k2} is positive constant and ρ_{k2} is a nonnegative smooth function.

Using (16), after simple calculations, it is no hard to obtain that

$$\begin{aligned}\frac{\partial({x_k^*}^{\frac{\sigma}{r_k}})}{\partial x_i} &= -\sum_{l=1}^{k-1}\frac{\partial B_l}{\partial x_i}x_l^{\frac{\sigma}{r_l}} - \frac{\sigma}{r_i}B_i x_i^{\frac{\sigma-r_i}{r_i}},\\ \frac{\partial^2({x_k^*}^{\frac{\sigma}{r_k}})}{\partial x_i^2} &= -\sum_{l=1}^{k-1}\frac{\partial^2 B_l}{\partial x_i^2}x_l^{\frac{\sigma}{r_l}} - \frac{\sigma}{r_i}\frac{\partial B_i}{\partial x_i}x_i^{\frac{\sigma-r_i}{r_i}}\\ &\quad -\frac{\sigma}{r_i}\frac{\partial B_i}{\partial x_i}x_i^{\frac{\sigma-r_i}{r_i}} - \frac{\sigma}{r_i}\cdot\frac{\sigma-r_i}{r_i}B_i x_i^{\frac{\sigma-2r_i}{r_i}},\\ \frac{\partial^2({x_k^*}^{\frac{\sigma}{r_k}})}{\partial x_i\partial x_j} &= -\sum_{l=1}^{k-1}\frac{\partial^2 B_l}{\partial x_i\partial x_j}x_l^{\frac{\sigma}{r_l}} - \frac{\sigma}{r_j}\frac{\partial B_j}{\partial x_i}x_j^{\frac{\sigma-r_j}{r_j}}\\ &\quad -\frac{\sigma}{r_i}\frac{\partial B_i}{\partial x_j}x_i^{\frac{\sigma-r_i}{r_i}},\end{aligned} \quad (A5)$$

for $i,j=1,\cdots,k-1$, $i\neq j$.

By (12),(17), (A3),(A5) and Lemma 3, we get

$$\begin{aligned}&\sum_{i=1}^{k-1}\frac{\partial W_k}{\partial x_i}(h_i x_{i+1}+f_i)\\ &= -\frac{4l\sigma-\tau-r_k}{\sigma}\int_{x_k^*}^{x_k}\left(s^{\frac{\sigma}{r_k}} - {x_k^*}^{\frac{\sigma}{r_k}}\right)^{\frac{(4l-1)\sigma-\tau-r_k}{\sigma}}ds\\ &\quad\times\sum_{i=1}^{k-1}\frac{\partial({x_k^*}^{\frac{\sigma}{r_k}})}{\partial x_i}(h_i x_{i+1}+f_i)\\ &\le c_k|\xi_k|^{\frac{(4l-1)\sigma-\tau}{\sigma}}\sum_{i=1}^{k-1}\Big(\sum_{q=1}^{k-1}\Big|\frac{\partial B_q}{\partial x_i}\Big||x_q|^{\frac{\sigma}{r_q}}\\ &\quad+\frac{\sigma}{r_i}|B_i||x_i|^{\frac{\sigma-r_i}{r_i}}\Big)(h_{i2}|x_{i+1}|+|f_i|)\\ &\le c_k\sum_{i=1}^{k-1}\sum_{q=1}^{k-1}\Big|\frac{\partial B_q}{\partial x_i}\Big||\xi_k|^{\frac{(4l-1)\sigma-\tau}{\sigma}}(|\xi_q|+\beta_{q-1}|\xi_{q-1}|)\\ &\quad\times\Big(h_{i2}(|\xi_{i+1}|+\beta_i|\xi_i|)^{\frac{r_i+\tau}{\sigma}}+\bar\varphi_i\sum_{m=1}^{i}|\xi_m|^{\frac{r_i+\tau}{\sigma}}\Big)\\ &\quad+c_k\sum_{i=1}^{k-1}\frac{\sigma}{r_i}|B_i||\xi_k|^{\frac{(4l-1)\sigma-\tau}{\sigma}}(|\xi_i|+\beta_{i-1}|\xi_{i-1}|)^{\frac{\sigma-r_i}{\sigma}}\\ &\quad\times\Big(h_{i2}(|\xi_{i+1}|+\beta_i|\xi_i|)^{\frac{r_i+\tau}{\sigma}}+\bar\varphi_i\sum_{m=1}^{i}|\xi_m|^{\frac{r_i+\tau}{\sigma}}\Big),\end{aligned} \quad (A6)$$

where $c_k>0$ is a constant.

Noting that $r_i>0$, by using Lemma 4, we have

$$\sum_{i=1}^{k-1}\frac{\partial W_k}{\partial x_i}(x_{i+1}+f_i) \le l_{k3}\sum_{i=1}^{k-1}\xi_i^{4l}+\xi_k^{4l}\rho_{k3}. \quad (A7)$$

where l_{k3} is positive constant and ρ_{k3} is a nonnegative smooth function.

From (5), (12) and Lemma 3, it follows that

$$\begin{aligned}|g_i| &\le \phi_i\sum_{j=1}^{i}|x_j|^{\frac{2r_i+\tau}{2r_j}}\\ &\le \phi_i\sum_{j=1}^{i}\Big(|\xi_j|+\beta_{j-1}|\xi_{j-1}|\Big)^{\frac{2r_i+\tau}{2\sigma}}\\ &\le \bar\phi_i\sum_{j=1}^{i}|\xi_j|^{\frac{2r_i+\tau}{2\sigma}},\end{aligned} \quad (A8)$$

where $\beta_0=\xi_0=0$ and $\bar\phi_i=\sum_{j=1}^{i}(1+\beta_{j-1}^{\frac{2r_i+\tau}{2\sigma}})\phi_i \ge 0$ is a smooth function.

According to (12), (17),(A5) and (A8), we have

$$\begin{aligned}&\frac{1}{2}\sum_{i,j=1,i\neq j}^{k-1}\Big|\frac{\partial^2 W_k}{\partial x_i\partial x_j}\Big||g_i^T g_j|\\ &=\frac{1}{2}\sum_{i,j=1,i\neq j}^{k-1}\Big|\frac{4l\sigma-\tau-r_k}{\sigma}\cdot\frac{(4l-1)\sigma-\tau-r_k}{\sigma}\\ &\quad\times\frac{\partial({x_k^*}^{\frac{\sigma}{r_k}})}{\partial x_i}\frac{\partial({x_k^*}^{\frac{\sigma}{r_k}})}{\partial x_j}\\ &\quad\times\int_{x_k^*}^{x_k}\left(s^{\frac{\sigma}{r_k}}-{x_k^*}^{\frac{\sigma}{r_k}}\right)^{\frac{(4l-2)\sigma-\tau-r_k}{\sigma}}ds\Big||g_i^T g_j|\\ &\le d_k\sum_{i,j=1,i\neq j}^{k-1}|\xi_k|^{\frac{(4l-2)\sigma-\tau}{\sigma}}\\ &\quad\times\Big|\sum_{q=1}^{k-1}\frac{\partial B_q}{\partial x_i}x_q^{\frac{\sigma}{r_q}}+\frac{\sigma}{r_i}B_i x_i^{\frac{\sigma-r_i}{r_i}}\Big|\\ &\quad\times\Big|\sum_{m=1}^{k-1}\frac{\partial B_m}{\partial x_j}x_m^{\frac{\sigma}{r_m}}+\frac{\sigma}{r_j}B_j x_j^{\frac{\sigma-r_j}{r_j}}\Big||g_i^T g_j|\\ &\le d_k\sum_{i,j=1,i\neq j}^{k-1}|\xi_k|^{\frac{(4l-2)\sigma-\tau}{\sigma}}\Big(\sum_{l=1}^{k-1}\Big|\frac{\partial B_l}{\partial x_i}\Big|\\ &\quad\times(|\xi_l|+\beta_{l-1}|\xi_{l-1}|)+\frac{\sigma}{r_i}|B_i|(|\xi_i|\\ &\quad+\beta_{i-1}|\xi_{i-1}|)^{\frac{\sigma-r_i}{\sigma}}\Big)\\ &\quad\times\Big(\sum_{m=1}^{k-1}\Big|\frac{\partial B_m}{\partial x_j}\Big|(|\xi_m|+\beta_{m-1}|\xi_{m-1}|)\\ &\quad+\frac{\sigma}{r_j}|B_j|(|\xi_j|+\beta_{j-1}|\xi_{j-1}|)^{\frac{\sigma-r_j}{\sigma}}\Big)\\ &\quad\times\Big(\bar\phi_i\sum_{p=1}^{i}|\xi_p|^{\frac{2r_i+\tau}{2\sigma}}\Big)\Big(\bar\phi_j\sum_{q=1}^{j}|\xi_q|^{\frac{2r_j+\tau}{2\sigma}}\Big),\end{aligned} \quad (A9)$$

where $d_k>0$ is a constant.

Noting that $r_i>0$ and $r_j>0$, by using Lemma

4, we have

$$\frac{1}{2}\sum_{i,j=1,i\neq j}^{k-1}\Big|\frac{\partial^2 W_k}{\partial x_i \partial x_j}\Big||g_i^T g_j| \leq l_{k4}\sum_{i=1}^{k-1}\xi_i^{4l}+\xi_k^{4l}\rho_{k4}, \tag{A10}$$

where l_{k4} is positive constant and ρ_{k4} is a nonnegative smooth function.

Similarly, we can obtain

$$\frac{1}{2}\sum_{i=1}^{k-1}\Big|\frac{\partial^2 W_k}{\partial x_i^2}\Big||g_i|^2 \leq l_{k5}\sum_{i=1}^{k-1}\xi_i^{4l}+\xi_k^{4l}\rho_{k5}. \tag{A11}$$

$$\frac{1}{2}\sum_{i=1}^{k-1}\Big|\frac{\partial^2 W_k}{\partial x_k \partial x_i}\Big||g_k^T g_i| \leq l_{k6}\sum_{i=1}^{k-1}\xi_i^{4l}+\xi_k^{4l}\rho_{k6}. \tag{A12}$$

$$\frac{1}{2}\Big|\frac{\partial^2 W_k}{\partial x_k^2}\Big||g_k|^2 \leq l_{k7}\sum_{i=1}^{k-1}\xi_i^{4l}+\xi_k^{4l}\rho_{k7}. \tag{A13}$$

where l_{kj}, and ρ_{kj}, $j = 5, 6, 7$ are the appropriate positive constants and nonnegative smooth functions, respectively. So far, by choosing $l_k = \sum_{j=1}^{7} l_{kj}$ and $\rho_k = \sum_{j=1}^{7} l_{kj}$, the proof of Proposition 1 is completed.

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