Existence and Global Attractivity of Positive Periodic Solutions in Shifts $\delta_{\pm}$ for a Nonlinear Dynamic Equation with Feedback Control on Time Scales

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Abstract: This paper is concerned with a nonlinear dynamic equation with feedback control on time scales. Based on the theory of calculus on time scales, by using a fixed point theorem of strict-set-contraction, some criteria are established for the existence of positive periodic solutions in shifts $\delta_{\pm}$ of the system; then, by using some differential inequalities and constructing a suitable Lyapunov functional, sufficient conditions which guarantee the global attractivity of the system are obtained. Finally, two numerical examples are presented to illustrate the feasibility and effectiveness of the results.

Key–Words: positive periodic solution; global attractivity; shift operator; feedback control; time scale.

1 Introduction

In the real world, lots of systems are continuously disturbed by unpredictable forces which can result in changes in parameters. Of practical interest is the question of whether or not a system can withstand those unpredictable disturbances which persist for a finite period of time. In the language of control variables, we call the disturbance functions as control variables. During the last decade, different types of functional differential and difference equations with feedback control have been extensively studied; see, for example, [1-5] and the references therein.

However, in applications, there are many systems whose developing processes are both continuous and discrete. Hence, using the only differential equation or difference equation can't accurately describe the law of their developments; see, for example, [6,7]. Therefore, there is a need to establish correspondent dynamic models on new time scales.

The theory of calculus on time scales (see [8] and references cited therein) was initiated by Stefan Hilger [9] in order to unify continuous and discrete analysis. Therefore, the study of dynamic equations on time scales, which unifies differential, difference, h-difference, and q-differences equations and more, has received much attention; see [10-14].

The existence problem of periodic solutions is an important topic in qualitative analysis of functional dynamic equations. Up to now, there are only a few results concerning periodic solutions of dynamic equations with feedback control on time scales; see, for example, [15,16]. In these papers, authors considered the existence of periodic solutions for dynamic equations on time scales satisfying the condition "there exists a $\omega > 0$ such that $t \pm \omega \in \mathbb{T}, \forall t \in \mathbb{T}$." Under this condition all periodic time scales are unbounded above and below. However, there are many time scales such as $q^{\mathbb{Z}} = \{q^n : n \in \mathbb{Z}\} \cup \{0\}$ and $\sqrt{\mathbb{N}} = \{\sqrt{n} : n \in \mathbb{N}\}$ which do not satisfy the condition. Adıvar and Raffoul introduced a new periodicity concept on time scales which does not oblige the time scale to be closed under the operation $t \pm \omega$ for a fixed $\omega > 0$. They defined a new periodicity concept with the aid of shift operators $\delta_{\pm}$ which are first defined in [17] and then generalized in [18].

Recently, by using the cone theory techniques, many researchers studied the existence and multiplicity of positive periodic solutions in shifts $\delta_{\pm}$ for some nonlinear first-order functional dynamic equations on time scales; see [19-22].

However, to the best of our knowledge, there are few papers published on the existence and global attractivity of positive periodic solutions in shifts $\delta_{\pm}$ for nonlinear dynamic systems with feedback control on time scales.

Motivated by the above statements, in the present paper, based on the theory of calculus on time scales, we shall study a nonlinear first-order dynamic equation with feedback control on time scales as follows:

$$\begin{cases} x^{\Delta}(t) &= x(t)[r(t)-a(t)x(t) \\ &\quad -b(t)x(\sigma(t))-c(t)y(t)], \\ y^{\Delta}(t) &= -\eta(t)y(t)+g(t)x(t), \end{cases} \quad (1)$$

where $t \in \mathbb{T}$, $\mathbb{T} \subset \mathbb{R}$ be a periodic time scale in shifts $\delta_{\pm}$ with period $P \in [t_0, \infty)_{\mathbb{T}}$ and $t_0 \in \mathbb{T}$ is nonnegative and fixed. $r, a, b, c, \eta, g \in C(\mathbb{T}, (0, \infty))$ are Δ-periodic functions in shifts $\delta_{\pm}$ with period ω and $-\eta \in \mathcal{R}^+$.

The initial condition of system (1) in the form

$$x(t_0) = x_0,\ y(t_0) = y_0,\ t_0 \in \mathbb{T},\ x_0 > 0,\ y_0 > 0. \quad (2)$$

The aim of this paper is, by using a fixed point theorem of strict-set-contraction, to obtain sufficient conditions for the existence of positive periodic solutions in shifts $\delta_{\pm}$ of system (1); and by using some differential inequalities and constructing a suitable Lyapunov functional, to obtain sufficient conditions for the global attractivity of system (1).

For convenience, we introduce the notation

$$f^u = \sup_{t \in [t_0, \delta_+^{\omega}(t_0)]_{\mathbb{T}}} f(t),\ f^l = \inf_{t \in [t_0, \delta_+^{\omega}(t_0)]_{\mathbb{T}}} f(t),$$

where f is a positive and bounded function.

Throughout this paper, we assume that

(H_1) $\Theta := e_r(t_0, \delta_+^{\omega}(t_0)) < 1$;

(H_2) $\min\{r^l, a^l, b^l, c^l, \eta^l, g^l\} > 0$, $\max\{r^u, a^u, b^u, c^u, \eta^u, g^u\} < \infty$;

(H_3) $r^l > a^u M_1 + c^u M_2$, where $M_1 = \frac{r^u}{b^l},\ M_2 = \frac{g^u M_1}{\eta^l}$;

(H_4) $a^l - g^u > 0$;

(H_5) $\eta^l - c^u > 0$.

2 Preliminaries

Let $\mathbb{T}$ be a nonempty closed subset (time scale) of $\mathbb{R}$. The forward and backward jump operators $\sigma, \rho : \mathbb{T} \to \mathbb{T}$ and the graininess $\mu : \mathbb{T} \to \mathbb{R}^+$ are defined, respectively, by

$$\sigma(t) = \inf\{s \in \mathbb{T} : s > t\},$$
$$\rho(t) = \sup\{s \in \mathbb{T} : s < t\}$$
$$\mu(t) = \sigma(t) - t.$$

A point $t \in \mathbb{T}$ is called left-dense if $t > \inf \mathbb{T}$ and $\rho(t) = t$, left-scattered if $\rho(t) < t$, right-dense if $t < \sup \mathbb{T}$ and $\sigma(t) = t$, and right-scattered if $\sigma(t) > t$. If $\mathbb{T}$ has a left-scattered maximum m, then $\mathbb{T}^k = \mathbb{T}\backslash\{m\}$; otherwise $\mathbb{T}^k = \mathbb{T}$. If $\mathbb{T}$ has a right-scattered minimum m, then $\mathbb{T}_k = \mathbb{T}\backslash\{m\}$; otherwise $\mathbb{T}_k = \mathbb{T}$.

For the basic theories of calculus on time scales, see [8].

A function $p : \mathbb{T} \to \mathbb{R}$ is called regressive provided $1 + \mu(t)p(t) \neq 0$ for all $t \in \mathbb{T}^k$. The set of all regressive and rd-continuous functions $p : \mathbb{T} \to \mathbb{R}$ will be denoted by $\mathcal{R} = \mathcal{R}(\mathbb{T}, \mathbb{R})$. Define the set $\mathcal{R}^+ = \mathcal{R}^+(\mathbb{T}, \mathbb{R}) = \{p \in \mathcal{R} : 1 + \mu(t)p(t) > 0, \forall\, t \in \mathbb{T}\}$.

If r is a regressive function, then the generalized exponential function e_r is defined by

$$e_r(t, s) = \exp\left\{ \int_s^t \xi_{\mu(\tau)}(r(\tau))\Delta\tau \right\}$$

for all $s, t \in \mathbb{T}$, with the cylinder transformation

$$\xi_h(z) = \begin{cases} \frac{\mathrm{Log}(1+hz)}{h}, & \text{if } h \neq 0, \\ z, & \text{if } h = 0. \end{cases}$$

Let $p, q : \mathbb{T} \to \mathbb{R}$ be two regressive functions, define

$$p \oplus q = p + q + \mu pq,\ \ominus p = -\frac{p}{1 + \mu p},\ p \ominus q = p \oplus (\ominus q).$$

Lemma 1. (see [8]) *If $p, q : \mathbb{T} \to \mathbb{R}$ be two regressive functions, then*
(i) $e_0(t, s) \equiv 1$ and $e_p(t, t) \equiv 1$;
(ii) $e_p(\sigma(t), s) = (1 + \mu(t)p(t))e_p(t, s)$;
(iii) $e_p(t, s) = \frac{1}{e_p(s,t)} = e_{\ominus p}(s, t)$;
(iv) $e_p(t, s)e_p(s, r) = e_p(t, r)$;
(v) $\frac{e_p(t,s)}{e_q(t,s)} = e_{p \ominus q}(t, s)$;
(vi) $(e_p(t, s))^{\Delta} = p(t)e_p(t, s)$.

The following definitions, lemmas about the shift operators and the new periodicity concept for time scales which can be found in [20,23].

Let $\mathbb{T}^*$ be a non-empty subset of the time scale $\mathbb{T}$ and $t_0 \in \mathbb{T}^*$ be a fixed number, define operators $\delta_{\pm} : [t_0, \infty) \times \mathbb{T}^* \to \mathbb{T}^*$. The operators δ_+ and δ_- associated with $t_0 \in \mathbb{T}^*$ (called the initial point) are said to be forward and backward shift operators on the set $\mathbb{T}^*$, respectively. The variable $s \in [t_0, \infty)_{\mathbb{T}}$ in $\delta_{\pm}(s, t)$ is called the shift size. The value $\delta_+(s, t)$ and $\delta_-(s, t)$ in $\mathbb{T}^*$ indicate s units translation of the term $t \in \mathbb{T}^*$ to the right and left, respectively. The sets

$$\mathbb{D}_{\pm} := \{(s, t) \in [t_0, \infty)_{\mathbb{T}} \times \mathbb{T}^* : \delta_{\mp}(s, t) \in \mathbb{T}^*\}$$

are the domains of the shift operator $\delta_{\pm}$, respectively. Hereafter, $\mathbb{T}^*$ is the largest subset of the time scale $\mathbb{T}$ such that the shift operators $\delta_{\pm} : [t_0, \infty) \times \mathbb{T}^* \to \mathbb{T}^*$ exist.

Definition 2. [23] *(Periodicity in shifts $\delta_\pm$) Let $\mathbb{T}$ be a time scale with the shift operators $\delta_\pm$ associated with the initial point $t_0 \in \mathbb{T}^*$. The time scale $\mathbb{T}$ is said to be periodic in shifts $\delta_\pm$ if there exists $p \in (t_0, \infty)_{\mathbb{T}^*}$ such that $(p, t) \in \mathbb{D}_\pm$ for all $t \in \mathbb{T}^*$. Furthermore, if*

$$P := \inf\{p \in (t_0, \infty)_{\mathbb{T}^*} : (p, t) \in \delta_\pm, \forall t \in \mathbb{T}^*\} \neq t_0,$$

then P is called the period of the time scale $\mathbb{T}$.

Definition 3. [23] *(Periodic function in shifts $\delta_\pm$) Let $\mathbb{T}$ be a time scale that is periodic in shifts $\delta_\pm$ with the period P. We say that a real-valued function f defined on $\mathbb{T}^*$ is periodic in shifts $\delta_\pm$ if there exists $\omega \in [P, \infty)_{\mathbb{T}^*}$ such that $(\omega, t) \in \mathbb{D}_\pm$ and $f(\delta^\omega_\pm(t)) = f(t)$ for all $t \in \mathbb{T}^*$, where $\delta^\omega_\pm := \delta_\pm(\omega, t)$. The smallest number $\omega \in [P, \infty)_{\mathbb{T}^*}$ is called the period of f.*

Definition 4. [23] *(Δ-periodic function in shifts $\delta_\pm$) Let $\mathbb{T}$ be a time scale that is periodic in shifts $\delta_\pm$ with the period P. We say that a real-valued function f defined on $\mathbb{T}^*$ is Δ-periodic in shifts $\delta_\pm$ if there exists $\omega \in [P, \infty)_{\mathbb{T}^*}$ such that $(\omega, t) \in \mathbb{D}_\pm$ for all $t \in \mathbb{T}^*$, the shifts $\delta^\omega_\pm$ are Δ-differentiable with rd-continuous derivatives and $f(\delta^\omega_\pm(t))\delta^{\Delta\omega}_\pm(t) = f(t)$ for all $t \in \mathbb{T}^*$, where $\delta^\omega_\pm := \delta_\pm(\omega, t)$. The smallest number $\omega \in [P, \infty)_{\mathbb{T}^*}$ is called the period of f.*

Lemma 5. [23] *$\delta^\omega_+(\sigma(t)) = \sigma(\delta^\omega_+(t))$ and $\delta^\omega_-(\sigma(t)) = \sigma(\delta^\omega_-(t))$ for all $t \in \mathbb{T}^*$.*

Lemma 6. [20] *Let $\mathbb{T}$ be a time scale that is periodic in shifts $\delta_\pm$ with the period P. Suppose that the shifts $\delta^\omega_\pm$ are Δ-differentiable on $t \in \mathbb{T}^*$ where $\omega \in [P, \infty)_{\mathbb{T}^*}$ and $p \in \mathcal{R}$ is Δ-periodic in shifts $\delta_\pm$ with the period ω. Then*

(i) $e_p(\delta^\omega_\pm(t), \delta^\omega_\pm(t_0)) = e_p(t, t_0)$ for $t, t_0 \in \mathbb{T}^*$;

(ii) $e_p(\delta^\omega_\pm(t), \sigma(\delta^\omega_\pm(s))) = e_p(t, \sigma(s)) = \frac{e_p(t,s)}{1+\mu(t)p(t)}$ for $t, s \in \mathbb{T}^*$.

Lemma 7. [23] *Let $\mathbb{T}$ be a time scale that is periodic in shifts $\delta_\pm$ with the period P, and let f be a Δ-periodic function in shifts $\delta_\pm$ with the period $\omega \in [P, \infty)_{\mathbb{T}^*}$. Suppose that $f \in C_{rd}(\mathbb{T})$, then*

$$\int_{t_0}^{t} f(s)\Delta s = \int_{\delta^\omega_\pm(t_0)}^{\delta^\omega_\pm(t)} f(s)\Delta s.$$

Lemma 8. [8] *Suppose that r is regressive and $f : \mathbb{T} \to \mathbb{R}$ is rd-continuous. Let $t_0 \in \mathbb{T}$, $y_0 \in \mathbb{R}$, then the unique solution of the initial value problem*

$$y^\Delta = r(t)y + f(t),\ y(t_0) = y_0$$

is given by

$$y(t) = e_r(t, t_0)y_0 + \int_{t_0}^{t} e_r(t, \sigma(\tau))f(\tau)\Delta\tau.$$

Lemma 9. (see [11]) *Assume that $a > 0, b > 0$ and $-a \in \mathcal{R}^+$. Then*

$$y^\Delta(t) \geq (\leq) b - ay(t),\ y(t) > 0, t \in [t_0, \infty)_\mathbb{T}$$

implies

$$y(t) \geq (\leq) \frac{b}{a}[1+(\frac{ay(t_0)}{b}-1)e_{(-a)}(t, t_0)], t \in [t_0, \infty)_\mathbb{T}.$$

Lemma 10. (see [11]) *Assume that $a > 0, b > 0$. Then*

$$y^\Delta(t) \leq (\geq) y(t)(b-ay(\sigma(t))),\ y(t) > 0, t \in [t_0, \infty)_\mathbb{T}$$

implies

$$y(t) \leq (\geq) \frac{b}{a}[1+(\frac{b}{ay(t_0)}-1)e_{\ominus b}(t, t_0)], t \in [t_0, \infty)_\mathbb{T}.$$

3 Existence results

In this section, we shall study the existence of at least one positive periodic solutions in shifts $\delta_\pm$ of system (1) via a fixed point theorem of strict-set-contraction.

Set

$$X = \{x : x \in C(\mathbb{T}, \mathbb{R}), x(\delta^\omega_+(t)) = x(t)\}$$

with the norm defined by $|x|_0 = \sup_{t \in [t_0, \delta^\omega_+(t_0)]_\mathbb{T}} |x(t)|$, then X is a Banach space.

By using Lemmas 1, 5 and 8, we can obtain the following lemma.

Lemma 11. *$x(t) \in X$ is an ω-periodic solution in shifts $\delta_\pm$ of system (1) if and only if $x(t)$ is an ω-periodic solution in shifts $\delta_\pm$ of*

$$x(t) = \int_{t}^{\delta^\omega_+(t)} G(t, s)x(s)\Bigg[a(s)x(s) + b(s)x(\sigma(s)) + c(s)\int_{s}^{\delta^\omega_+(s)} H(s, \theta)g(\theta)x(\theta)\Delta\theta\Bigg]\Delta s, \quad (3)$$

where

$$G(t, s) = \frac{e_r(t, \sigma(s))}{1 - e_r(t_0, \delta^\omega_+(t_0))},$$

and

$$H(t, s) = \frac{e_{-\eta}(t, \sigma(s))}{e_{-\eta}(t_0, \delta^\omega_+(t_0)) - 1}.$$

Proof. If $y(t)$ is an ω-periodic solution in shifts $\delta_{\pm}$ of the second equation of system (1). By using Lemma 8, for $s \in [t, \delta_{+}^{\omega}(t)]_{\mathbb{T}}$, we have

$$y(s) = e_{-\eta}(s,t)y(t) + \int_t^s e_{-\eta}(s,\sigma(\theta))g(\theta)x(\theta)\Delta\theta.$$

Let $s = \delta_{+}^{\omega}(t)$ in the above equality, we have

$$\begin{aligned} y(\delta_{+}^{\omega}(t)) &= e_{-\eta}(\delta_{+}^{\omega}(t),t)y(t) \\ &\quad + \int_t^{\delta_{+}^{\omega}(t)} e_{-\eta}(\delta_{+}^{\omega}(t),\sigma(\theta))g(\theta)x(\theta)\Delta\theta. \end{aligned}$$

Noticing that $y(\delta_{+}^{\omega}(t)) = y(t)$ and $e_{-\eta}(t,\delta_{+}^{\omega}(t)) = e_{-\eta}(t_0,\delta_{+}^{\omega}(t_0))$, then

$$y(t) = \int_t^{\delta_{+}^{\omega}(t)} H(t,s)g(s)x(s)\Delta s := (\Psi x)(t), \quad (4)$$

where

$$H(t,s) = \frac{e_{-\eta}(t,\sigma(s))}{e_{-\eta}(t_0,\delta_{+}^{\omega}(t_0)) - 1}.$$

Let $y(t)$ be an ω-periodic solution in shifts $\delta_{\pm}$ of (4). By Lemmas 1 and 5, we have

$$\begin{aligned} y^{\Delta}(t) &= -\eta(t)y(t) \\ &\quad + H(\sigma(t),\delta_{+}^{\omega}(t))g(\delta_{+}^{\omega}(t))\delta_{+}^{\Delta\omega}(t)x(\delta_{+}^{\omega}(t)) \\ &\quad - H(\sigma(t),t)g(t)x(t) \\ &= -\eta(t)y(t) + g(t)x(t). \end{aligned}$$

Therefore, the existence problem of ω-periodic solutions in shifts $\delta_{\pm}$ of system (1) is equivalent to that of the following equation

$$\begin{aligned} x^{\Delta}(t) &= x(t)[r(t) - a(t)x(t) - b(t)x(\sigma(t)) \\ &\quad - c(t)\int_t^{\delta_{+}^{\omega}(t)} H(t,s)g(s)x(s)\Delta s]. \quad (5) \end{aligned}$$

Repeating the above process, it follows from (5) that $x(t) \in X$ is an ω-periodic solution in shifts $\delta_{\pm}$ of system (1) if and only if $x(t)$ is an ω-periodic solution in shifts $\delta_{\pm}$ of

$$\begin{aligned} x(t) &= \int_t^{\delta_{+}^{\omega}(t)} G(t,s)x(s)\Bigg[a(s)x(s) + b(s)x(\sigma(s)) \\ &\quad + c(s)\int_s^{\delta_{+}^{\omega}(s)} H(s,\theta)g(\theta)x(\theta)\Delta\theta\Bigg]\Delta s, \end{aligned}$$

where

$$G(t,s) = \frac{e_r(t,\sigma(s))}{1 - e_r(t_0,\delta_{+}^{\omega}(t_0))}.$$

This proof is complete. □

It is easy to verify that the Green's functions $G(t,s)$ and $H(t,s)$ satisfy the property

$$0 < \frac{\Theta}{1-\Theta} \le G(t,s) \le \frac{1}{1-\Theta}, \ \forall s \in [t,\delta_{+}^{\omega}(t)]_{\mathbb{T}}, \quad (6)$$

and

$$0 < \frac{1}{\Gamma - 1} \le H(t,s) \le \frac{\Gamma}{\Gamma - 1}.$$

where $\Theta := e_r(t_0,\delta_{+}^{\omega}(t_0)), \Gamma := e_{-\eta}(t_0,\delta_{+}^{\omega}(t_0))$. By Lemma 6, we have

$$\begin{aligned} &G(\delta_{+}^{\omega}(t),\delta_{+}^{\omega}(s)) = G(t,s), \\ &H(\delta_{+}^{\omega}(t),\delta_{+}^{\omega}(s)) = H(t,s), \\ &\forall t \in \mathbb{T}^*, s \in [t,\delta_{+}^{\omega}(t)]_{\mathbb{T}}. \end{aligned} \quad (7)$$

In order to obtain the existence of periodic solutions in shifts $\delta_{\pm}$ of system (1), we first make the following preparations:

Let E be a Banach space and K be a cone in E. The semi-order induced by the cone K is denoted by "$\le$", that is, $x \le y$ if and only if $y - x \in K$. In addition, for a bounded subset $A \subset E$, let $\alpha_E(A)$ denote the (Kuratowski) measure of non-compactness defined by

$$\begin{aligned} \alpha_E(A) = \inf\big\{&d > 0 : \text{there is a finite number of} \\ &\text{subsets } A_i \subset A \text{ such that } A = \bigcup_i A_i \\ &\text{and } \operatorname{diam}(A_i) \le d\big\}, \end{aligned}$$

where $\operatorname{diam}(A_i)$ denotes the diameter of the set A_i.

Let E, F be two Banach spaces and $D \subset E$, a continuous and bounded map $\Phi : \bar{\Omega} \to F$ is called k-set contractive if for any bounded set $S \subset D$ we have

$$\alpha_F(\Phi(S)) \le k\alpha_E(S).$$

Φ is called strict-set-contractive if it is k-set-contractive for some $0 \le k < 1$.

Lemma 12. [24, 25] *Let K be a cone of the real Banach space X and $K_{r,R} = \{x \in K | r \le x \le R\}$ with $R > r > 0$. Suppose that $\Phi : K_{r,R} \to K$ is strict-set-contractive such that one of the following two conditions is satisfied:*

(*i*) $\Phi x \not\le x, \ \forall x \in K, \ ||x|| = r$ *and* $\Phi x \not\ge x, \ \forall x \in K, \ ||x|| = R$.

(*ii*) $\Phi x \not\ge x, \ \forall x \in K, \ ||x|| = r$ *and* $\Phi x \not\le x, \ \forall x \in K, \ ||x|| = R$.

Then Φ has at least one fixed point in $K_{r,R}$.

Define K, a cone in X, by

$$K = \{x \in X : x(t) \geq \Theta |x|_0, t \in [t_0, \delta_+^\omega(t_0)]_{\mathbb{T}}\}, \tag{8}$$

and an operator $\Phi : K \to X$ by

$$\begin{aligned}&(\Phi x)(t)\\ =\ & \int_t^{\delta_+^\omega(t)} G(t,s)x(s)\Big[a(s)x(s) + b(s)x(\sigma(s))\\ &+c(s)\int_s^{\delta_+^\omega(s)} H(s,\theta)g(\theta)x(\theta)\Delta\theta\Big]\Delta s. \end{aligned}\tag{9}$$

In the following, we shall give some lemmas concerning K and Φ defined by (8) and (9), respectively.

Lemma 13. *Assume that (H_1) holds, then $\Phi : K \to K$ is well defined.*

Proof. For any $x \in K$. In view of (4), for $t \in \mathbb{T}$, we obtain

$$\begin{aligned}&(\Psi x)(\delta_+^\omega(t))\\ =\ & \int_{\delta_+^\omega(t)}^{\delta_+^\omega(\delta_+^\omega(t))} H(\delta_+^\omega(t), s)g(s)x(s)\Delta s\\ =\ & \int_t^{\delta_+^\omega(t)} H(\delta_+^\omega(t), \delta_+^\omega(u))g(\delta_+^\omega(u))\delta_+^{\Delta\omega}(u)\\ &\times x(\delta_+^\omega(u))\Delta u\\ =\ & \int_t^{\delta_+^\omega(t)} H(t,u)g(u)x(u)\Delta u\\ =\ & (\Psi x)(t),\end{aligned}$$

then

$$\begin{aligned}&(\Phi x)(\delta_+^\omega(t))\\ =\ & \int_{\delta_+^\omega(t)}^{\delta_+^\omega(\delta_+^\omega(t))} G(\delta_+^\omega(t), s)x(s)\Big[a(s)x(s)\\ &+b(s)x(\sigma(s)) + c(s)(\Psi x)(s)\Big]\Delta s\\ =\ & \int_t^{\delta_+^\omega(t)} G(\delta_+^\omega(t), \delta_+^\omega(u))x(\delta_+^\omega(u))\\ &\times\Big[a(\delta_+^\omega(u))\delta_+^{\Delta\omega}(u)x(\delta_+^\omega(u))\\ &+b(\delta_+^\omega(u))\delta_+^{\Delta\omega}(u)x(\sigma(\delta_+^\omega(u)))\\ &+c(\delta_+^\omega(u))\delta_+^{\Delta\omega}(u)(\Psi x)(\delta_+^\omega(u))\Big]\Delta u\\ =\ & \int_t^{\delta_+^\omega(t)} G(t,u)x(u)\Big[a(u)x(u)\\ &+b(u)x(\sigma(u)) + c(u)(\Psi x)(u)\Big]\Delta u\\ =\ & (\Phi x)(t),\end{aligned}$$

that is, $(\Phi x)(\delta_+^\omega(t)) = (\Phi x)(t), t \in \mathbb{T}$. So $\Phi x \in X$.

Furthermore, for $x \in K, t \in [t_0, \delta_+^\omega(t_0)]_{\mathbb{T}}$, we have

$$\begin{aligned}&|\Phi x|_0\\ \leq\ & \frac{1}{1-\Theta}\int_{t_0}^{\delta_+^\omega(t_0)} x(s)\Big[a(s)x(s) + b(s)x(\sigma(s))\\ &+c(s)\int_s^{\delta_+^\omega(s)} H(s,\theta)g(\theta)x(\theta)\Delta\theta\Big]\Delta s\end{aligned}$$

and

$$\begin{aligned}&(\Phi x)(t)\\ \geq\ & \frac{\Theta}{1-\Theta}\int_{t_0}^{\delta_+^\omega(t_0)} x(s)\Big[a(s)x(s) + b(s)x(\sigma(s))\\ &+c(s)\int_s^{\delta_+^\omega(s)} H(s,\theta)g(\theta)x(\theta)\Delta\theta\Big]\Delta s\\ =\ & \Theta\frac{1}{1-\Theta}\int_{t_0}^{\delta_+^\omega(t_0)} x(s)\Big[a(s)x(s) + b(s)x(\sigma(s))\\ &+c(s)\int_s^{\delta_+^\omega(s)} H(s,\theta)g(\theta)x(\theta)\Delta\theta\Big]\Delta s\\ \geq\ & \Theta|\Phi x|_0,\end{aligned}\tag{10}$$

that is, $\Phi x \in K$. The proof is complete. □

Lemma 14. *Assume that $(H_1) - (H_2)$ hold, then $\Phi : K\bigcap\bar{\Omega}_R \to K$ is strict-set-contractive, where $\Omega_R = \{x \in X : |x|_0 < R\}$.*

Proof. It is easy to see that Φ is continuous and bounded. Now we prove that $\alpha_X(\Phi(S)) \leq k\alpha_X(S)$ for any bounded set $S \subset \bar{\Omega}_R$ and $0 < k < 1$. Let $\eta = \alpha_X(S)$. Then, for any positive number $\varepsilon < \eta$, there is a finite family of subsets $\{S_i\}$ satisfying $S = \bigcup_i S_i$ with diam$(S_i) \leq \eta + \varepsilon$. Therefore

$$|x - y|_0 \leq \eta + \varepsilon \quad \text{for any } x, y \in S_i. \tag{11}$$

As S and S_i are precompact in X, it follows that there is a finite family of subsets $\{S_{ij}\}$ of S_i such that $S_i = \bigcup_j S_{ij}$ and

$$|x - y|_0 \leq \varepsilon \quad \text{for any } x, y \in S_{ij}. \tag{12}$$

In addition, for any $x \in S$ and $t \in [t_0, \delta_+^\omega(t_0)]_{\mathbb{T}}$, we have

$$\begin{aligned}&|(\Phi x)(t)|\\ =\ & \int_t^{\delta_+^\omega(t)} G(t,s)x(s)\Big[a(s)x(s) + b(s)x(\sigma(s))\\ &+c(s)\int_s^{\delta_+^\omega(s)} H(s,\theta)g(\theta)x(\theta)\Delta\theta\Big]\Delta s\\ \leq\ & \frac{R^2}{1-\Theta}\Pi := A,\end{aligned}$$

where $\Pi := \int_{t_0}^{\delta_+^\omega(t_0)}\Big[a(s)+b(s)+c(s)\int_s^{\delta_+^\omega(s)} H(s,\theta)$ $g(\theta)\Delta\theta\Big]\Delta s$. And

$$\begin{aligned} & |(\Phi x)^\Delta(t)| \\ = & \Big|r(t)(\Phi x)(t) - x(t)\Big[a(t)x(t) + b(t)x(\sigma(t)) \\ & +c(t)\int_t^{\delta_+^\omega(t)} H(t,\theta)g(\theta)x(\theta)\Delta\theta\Big]\Big| \\ \le & \; r^u A \\ & +R^2\Big(a^u + b^u + c^u\frac{\Gamma}{\Gamma-1}\int_{t_0}^{\delta_+^\omega(t_0)} g(\theta)\Delta\theta\Big). \end{aligned}$$

Applying the Arzela-Ascoli Theorem, we know that $\Phi(S)$ is precompact in X. Then, there is a finite family of subsets $\{S_{ijk}\}$ of S_{ij} such that $S_{ij} = \bigcup_k S_{ijk}$ and

$$|\Phi x - \Phi y|_0 \le \varepsilon \quad \text{for any } x, y \in S_{ijk}. \tag{13}$$

As ε is arbitrary small, it follows that

$$\alpha_X(\Phi(S)) \le k\alpha_X(S).$$

Therefore, Φ is strict-set-contractive. The proof is complete. □

Theorem 15. *Assume that $(H_1)-(H_2)$ hold, then system (1) has at least one positive ω-periodic solution in shifts $\delta_\pm$.*

Proof. Let $R = \frac{1-\Theta}{\Theta^3\Pi}$ and $0 < r < \frac{\Theta(1-\Theta)}{\Pi}$, then we have $0 < r < R$. From Lemmas 13 and 14, we know that Φ is strict-set-contractive on $K_{r,R}$. In view of (9), we see that if there exists $x^* \in K$ such that $\Phi x^* = x^*$, then x^* is one positive ω-periodic solution in shifts $\delta_\pm$ of system (1). Now, we shall prove that condition (ii) of Lemma 12 holds.

First, we prove that $\Phi x \ngeq x$, $\forall x \in K$, $|x|_0 = r$. Otherwise, there exists $x \in K$, $|x|_0 = r$ such that $\Phi x \ge x$. So $|x| > 0$ and $\Phi x - x \in K$, which implies that

$$(\Phi x)(t) - x(t) \ge \Theta|\Phi x - x|_0 \ge 0, \forall t \in [t_0, \delta_+^\omega(t_0)]_{\mathbb{T}}. \tag{14}$$

Moreover, for $t \in [t_0, \delta_+^\omega(t_0)]_{\mathbb{T}}$, we have

$$\begin{aligned} & (\Phi x)(t) \\ = & \int_t^{\delta_+^\omega(t)} G(t,s)x(s)\Big[a(s)x(s) + b(s)x(\sigma(s)) \\ & +c(s)\int_s^{\delta_+^\omega(s)} H(s,\theta)g(\theta)x(\theta)\Delta\theta\Big]\Delta s \\ \le & \frac{1}{1-\Theta} r|x|_0 \int_{t_0}^{\delta_+^\omega(t_0)}\Big[a(s) + b(s) \\ & +c(s)\int_s^{\delta_+^\omega(s)} H(s,\theta)g(\theta)\Delta\theta\Big]\Delta s \\ = & \frac{r}{1-\Theta}\Pi|x|_0 \\ < & \Theta|x|_0. \end{aligned} \tag{15}$$

In view of (14) and (15), we have

$$|x|_0 \le |\Phi x| < \Theta|x|_0 < |x|_0,$$

which is a contradiction. Finally, we prove that $\Phi x \nleq x$, $\forall x \in K$, $|x|_0 = R$ also holds. For this case, we only need to prove that

$$\Phi x \not< x \quad x \in K, |x|_0 = R.$$

Suppose, for the sake of contradiction, that there exists $x \in K$ and $|x|_0 = R$ such that $\Phi x < x$. Thus $x - \Phi x \in K \setminus \{0\}$. Furthermore, for any $t \in [t_0, \delta_+^\omega(t_0)]_{\mathbb{T}}$, we have

$$x(t) - (\Phi x)(t) \ge \Theta|x - \Phi x|_0 > 0. \tag{16}$$

In addition, for any $t \in [t_0, \delta_+^\omega(t_0)]_{\mathbb{T}}$, we find

$$\begin{aligned} & (\Phi x)(t) \\ = & \int_t^{\delta_+^\omega(t)} G(t,s)x(s)\Big[a(s)x(s) + b(s)x(\sigma(s)) \\ & +c(s)\int_s^{\delta_+^\omega(s)} H(s,\theta)g(\theta)x(\theta)\Delta\theta\Big]\Delta s \\ \ge & \frac{\Theta}{1-\Theta}\Theta^2|x|_0^2\int_{t_0}^{\delta_+^\omega(t_0)}\Big[a(s) + b(s) \\ & +c(s)\int_s^{\delta_+^\omega(s)} H(s,\theta)g(\theta)\Delta\theta\Big]\Delta s \\ = & \frac{\Theta^3}{1-\Theta}\Pi R^2 \\ = & R. \end{aligned} \tag{17}$$

From (16) and (17), we obtain

$$|x| > |\Phi x|_0 \ge R,$$

which is a contradiction. Therefore, conditions (i) and (ii) hold. By Lemma 12, we see that Φ has at least one nonzero fixed point in K. Therefore, system (1) has at least one positive ω-periodic solution in shifts $\delta_\pm$. The proof is complete. □

4 Global attractivity

In this section, we shall study the global attractivity of positive periodic solution in shifts $\delta_\pm$ of system (1) with initial condition (2). Applying Lemmas 9 and 10, we can obtain the following lemma.

Lemma 16. *Let $(x(t), y(t))$ be any positive periodic solution in shifts $\delta_\pm$ of system (1) with initial condition (2). If (H_3) hold, then system (1) is permanent, that is, any positive periodic solution in shifts $\delta_\pm$ $(x(t), y(t))$ of system (1) satisfies*

$$m_1 \le \liminf_{t\to\infty} x(t) \le \limsup_{t\to\infty} x(t) \le M_1, \quad (18)$$
$$m_2 \le \liminf_{t\to\infty} y(t) \le \limsup_{t\to\infty} y(t) \le M_2, \quad (19)$$

especially if $m_1 \le x_0 \le M_1$, $m_2 \le y_0 \le M_2$, then

$$m_1 \le x(t) \le M_1,\ m_2 \le y(t) \le M_2,\ t \in [t_0, \infty)_{\mathbb{T}},$$

where

$$M_1 = \frac{r^u}{b^l},\ M_2 = \frac{g^u M_1}{\eta^l},$$
$$m_1 = \frac{r^l - a^u M_1 - c^u M_2}{b^u},\ m_2 = \frac{g^l m_1}{\eta^u}.$$

Lemma 17. *Assume that $(H_3) - (H_5)$ hold, then system (1) is globally attractive.*

Proof. Let $z_1(t) = (x_1(t), y_1(t))$ and $z_2(t) = (x_2(t), y_2(t))$ be any two positive periodic solution in shifts $\delta_\pm$ of system (1). It follows from (18)-(19) that for sufficient small positive constant ε_0 $(0 < \varepsilon_0 < \min\{m_1, m_2\})$, there exists a $T > 0$ such that

$$\begin{aligned} m_1 - \varepsilon_0 < x_i(t) < M_1 + \varepsilon_0, \\ m_2 - \varepsilon_0 < y_i(t) < M_2 + \varepsilon_0, \end{aligned} \quad (20)$$

where $t \in [T, +\infty)_{\mathbb{T}}, i = 1, 2$.

Since $x_i(t), i = 1, 2$ are positive, bounded and differentiable functions on $\mathbb{T}$, then there exists a positive, bounded and differentiable function $m(t), t \in \mathbb{T}$, such that $x_i(t)(1+m(t)), i = 1, 2$ are strictly increasing on $\mathbb{T}$. Then

$$\begin{aligned} & \frac{\Delta}{\Delta t} L_{\mathbb{T}}(x_i(t)[1 + m(t)]) \\ = \ & \frac{x_i^\Delta(t)[1 + m(t)] + x_i(\sigma(t))m^\Delta(t)}{x_i(t)[1 + m(t)]} \\ = \ & \frac{x_i^\Delta(t)}{x_i(t)} + \frac{x_i(\sigma(t))m^\Delta(t)}{x_i(t)[1 + m(t)]},\ i = 1, 2. \end{aligned}$$

Here, we can choose a function $m(t)$ such that $\frac{|m^\Delta(t)|}{1+m(t)}$ is bounded on $\mathbb{T}$, that is, there exist two positive constants $\zeta > 0$ and $\xi > 0$ such that $0 < \zeta < \frac{|m^\Delta(t)|}{1+m(t)} < \xi$, $\forall t \in \mathbb{T}$.

Set

$$\begin{aligned} V(t) \ = \ & |e_{-\delta}(t, T)|(|L_{\mathbb{T}}(x_1(t)(1 + m(t))) \\ & - L_{\mathbb{T}}(x_2(t)(1 + m(t)))| \\ & + |y_1(t) - y_2(t)|), \end{aligned}$$

where $\delta \ge 0$ is a constant (if $\mu(t) = 0$, then $\delta = 0$; if $\mu(t) > 0$, then $\delta > 0$). It follows from the mean value theorem of differential calculus on time scales for $t \in [T, +\infty)_{\mathbb{T}}$,

$$\begin{aligned} & \frac{1}{M_1 + \varepsilon_0}|x_1(t) - x_2(t)| \\ \le \ & |L_{\mathbb{T}}(x_1(t)(1 + m(t))) - L_{\mathbb{T}}(x_2(t)(1 + m(t)))| \\ \le \ & \frac{1}{m_1 - \varepsilon_0}|x_1(t) - x_2(t)|. \end{aligned} \quad (21)$$

Let $\gamma = \min\{(m_1 - \varepsilon_0)(a^l - g^u), \eta^l - c^u\}$. We divide the proof into two cases.

Case I. If $\mu(t) > 0$, set $\delta > \max\{(b^u + \frac{\xi}{m_1})M_1, \gamma\}$ and $1 - \mu(t)\delta < 0$. Calculating the upper right derivatives of $V(t)$ along the solution of system (1), it follows from (20), (21), (H_4) and (H_5) that for $t \in [T, +\infty)_{\mathbb{T}}$,

$$\begin{aligned} & D^+V^\Delta(t) \\ = \ & |e_{-\delta}(t, T)|\mathrm{sgn}(x_1(t) - x_2(t))\bigg[\frac{x_1^\Delta(t)}{x_1(t)} - \frac{x_2^\Delta(t)}{x_2(t)} \\ & + \frac{m^\Delta(t)}{1 + m(t)}\bigg(\frac{x_1(\sigma(t))}{x_1(t)} - \frac{x_2(\sigma(t))}{x_2(t)}\bigg)\bigg] \\ & - \delta|e_{-\delta}(t, T)||L_{\mathbb{T}}(x_1(\sigma(t))(1 + m(\sigma(t)))) \\ & - L_{\mathbb{T}}(x_2(\sigma(t))(1 + m(\sigma(t))))| \\ & + |e_{-\delta}(t, T)|\mathrm{sgn}(y_1(t) - y_2(t))(y_1^\Delta(t) - y_2^\Delta(t)) \\ & - \delta|e_{-\delta}(t, T)||y_1(\sigma(t)) - y_2(\sigma(t))| \\ = \ & |e_{-\delta}(t, T)|\mathrm{sgn}(x_1(t) - x_2(t)) \\ & \times\bigg[- a(t)(x_1(t) - x_2(t)) \\ & - b(t)(x_1(\sigma(t)) - x_2(\sigma(t))) \\ & - c(t)(y_1(t) - y_2(t))] \\ & + \frac{m^\Delta(t)}{1 + m(t)}\frac{x_1(\sigma(t))x_2(t) - x_1(t)x_2(\sigma(t))}{x_1(t)x_2(t)}\bigg] \\ & - \delta|e_{-\delta}(t, T)||L_{\mathbb{T}}(x_1(\sigma(t))(1 + m(\sigma(t)))) \\ & - L_{\mathbb{T}}(x_2(\sigma(t))(1 + m(\sigma(t))))| \\ & + |e_{-\delta}(t, T)|\mathrm{sgn}(y_1(t) - y_2(t)) \\ & \times[-\eta(t)(y_1(t) - y_2(t)) + (x_1(t) - x_2(t))] \\ & - \delta|e_{-\delta}(t, T)||y_1(\sigma(t)) - y_2(\sigma(t))| \\ = \ & |e_{-\delta}(t, T)|\mathrm{sgn}(x_1(t) - x_2(t)) \\ & \times\bigg[- a(t)(x_1(t) - x_2(t)) \\ & - b(t)(x_1(\sigma(t)) - x_2(\sigma(t))) \\ & - c(t)(y_1(t) - y_2(t))] \\ & + \frac{m^\Delta(t)}{1 + m(t)}\bigg(\frac{x_1(\sigma(t))(x_2(t) - x_1(t))}{x_1(t)x_2(t)} \\ & + \frac{x_1(t)(x_1(\sigma(t)) - x_2(\sigma(t)))}{x_1(t)x_2(t)}\bigg)\bigg] \end{aligned}$$

$$
\begin{aligned}
& -\delta|e_{-\delta}(t,T)||L_{\mathbb{T}}(x_1(\sigma(t))(1+m(\sigma(t)))) \\
& -L_{\mathbb{T}}(x_2(\sigma(t))(1+m(\sigma(t))))| \\
& +|e_{-\delta}(t,T)|\mathrm{sgn}(y_1(t)-y_2(t)) \\
& \times[-\eta(t)(y_1(t)-y_2(t))+(x_1(t)-x_2(t))] \\
& -\delta|e_{-\delta}(t,T)||y_1(\sigma(t))-y_2(\sigma(t))| \\
\le\ & -|e_{-\delta}(t,T)|\bigg[a(t)-g(t) \\
& +\frac{|m^{\Delta}(t)|}{1+m(t)}\frac{x_1(\sigma(t))}{x_1(t)x_2(t)}\bigg]|x_1(t)-x_2(t)| \\
& -|e_{-\delta}(t,T)|\bigg[\frac{\delta}{M_1+\varepsilon_0}-b(t) \\
& -\frac{|m^{\Delta}(t)|}{1+m(t)}\frac{1}{x_2(t)}\bigg]|x_1(\sigma(t))-x_2(\sigma(t))| \\
& -|e_{-\delta}(t,T)|(\eta(t)-c(t))|y_1(t)-y_2(t)| \\
& -\delta|e_{-\delta}(t,T)||y_1(\sigma(t))-y_2(\sigma(t))| \\
\le\ & -|e_{-\delta}(t,T)|(a^l-g^u)|x_1(t)-x_2(t)| \\
& -|e_{-\delta}(t,T)|(\eta^l-c^u)|y_1(t)-y_2(t)| \\
\le\ & -|e_{-\delta}(t,T)|((m_1-\varepsilon_0)(a^l-g^u) \\
& \times|L_{\mathbb{T}}(x_1(t)(1+m(t))) \\
& -L_{\mathbb{T}}(x_2(t)(1+m(t)))| \\
& +(\eta^l-c^u)|y_1(t)-y_2(t)|) \\
\le\ & -\gamma|e_{-\delta}(t,T)|(|L_{\mathbb{T}}(x_1(t)(1+m(t))) \\
& -L_{\mathbb{T}}(x_2(t)(1+m(t)))|+|y_1(t)-y_2(t)|) \\
=\ & -\gamma V(t). \qquad (22)
\end{aligned}
$$

By the comparison theorem and (22), we have

$$
\begin{aligned}
& V(t)\le|e_{-\gamma}(t,T)|V(T) \\
& <2\bigg(\frac{M_1+\varepsilon_0}{m_1-\varepsilon_0}+M_2+\varepsilon_0\bigg)|e_{-\gamma}(t,T)|,
\end{aligned}
$$

that is,

$$
\begin{aligned}
& |e_{-\delta}(t,T)|(|L_{\mathbb{T}}(x_1(t)(1+m(t))) \\
& -L_{\mathbb{T}}(x_2(t)(1+m(t)))|+|y_1(t)-y_2(t)|) \\
& <2\bigg(\frac{M_1+\varepsilon_0}{m_1-\varepsilon_0}+M_2+\varepsilon_0\bigg)|e_{-\gamma}(t,T)|,
\end{aligned}
$$

then

$$
\begin{aligned}
& \frac{1}{M_1+\varepsilon_0}|x_1(t)-x_2(t)|+|y_1(t)-y_2(t)| \\
& <2\bigg(\frac{M_1+\varepsilon_0}{m_1-\varepsilon_0}+M_2+\varepsilon_0\bigg) \\
& \times|e_{(-\gamma)\ominus(-\delta)}(t,T)|. \qquad (23)
\end{aligned}
$$

Since $1-\mu(t)\delta<0$ and $0<\gamma<\delta$, then $(-\gamma)\ominus(-\delta)<0$. It follows from (23) that

$$\lim_{t\to+\infty}|x_1(t)-x_2(t)|=0,\ \lim_{t\to+\infty}|y_1(t)-y_2(t)|=0.$$

Case II. If $\mu(t)=0$, set $\delta=0$, then $\sigma(t)=t$ and $|e_{-\delta}(t,T)|=1$. Calculating the upper right derivatives of $V(t)$ along the solution of system (1), it follows from (20), (21), (H_4) and (H_5) that for $t\in[T,+\infty)_{\mathbb{T}}$,

$$
\begin{aligned}
& D^+V^{\Delta}(t) \\
=\ & \mathrm{sgn}(x_1(t)-x_2(t))\bigg(\frac{x_1^{\Delta}(t)}{x_1(t)}-\frac{x_2^{\Delta}(t)}{x_2(t)}\bigg) \\
& +\mathrm{sgn}(y_1(t)-y_2(t))(y_1^{\Delta}(t)-y_2^{\Delta}(t)) \\
=\ & \mathrm{sgn}(x_1(t)-x_2(t))[-(a(t)+b(t)) \\
& \times(x_1(t)-x_2(t))-c(t)(y_1(t)-y_2(t))] \\
& +\mathrm{sgn}(y_1(t)-y_2(t))[-\eta(t)(y_1(t)-y_2(t)) \\
& +g(t)(x_1(t)-x_2(t))] \\
\le\ & -(a(t)+b(t)-g(t))|x_1(t)-x_2(t)| \\
& -(\eta(t)-c(t))|y_1(t)-y_2(t)| \\
\le\ & -((m_1-\varepsilon_0)(a^l+b^l-g^u) \\
& \times|L_{\mathbb{T}}(x_1(t)(1+m(t))) \\
& -L_{\mathbb{T}}(x_2(t)(1+m(t)))| \\
& +(\eta^l-c^u)|y_1(t)-y_2(t)|) \\
\le\ & -\widehat{\gamma}(|L_{\mathbb{T}}(x_1(t)(1+m(t))) \\
& -L_{\mathbb{T}}(x_2(t)(1+m(t)))|+|y_1(t)-y_2(t)|) \\
\le\ & -\gamma V(t), \qquad (24)
\end{aligned}
$$

where $\widehat{\gamma}=\min\{(m_1-\varepsilon_0)(a^l+b^l-g^u),\eta^l-c^u\}$. By the comparison theorem and (24), we have

$$
\begin{aligned}
V(t)\ &\le\ |e_{-\gamma}(t,T)|V(T) \\
&<\ 2\bigg(\frac{M_1+\varepsilon_0}{m_1-\varepsilon_0}+M_2+\varepsilon_0\bigg)|e_{-\gamma}(t,T)|,
\end{aligned}
$$

that is,

$$
\begin{aligned}
& |L_{\mathbb{T}}(x_1(t)(1+m(t)))-L_{\mathbb{T}}(x_2(t)(1+m(t)))| \\
& +|y_1(t)-y_2(t)| \\
& <2\bigg(\frac{M_1+\varepsilon_0}{m_1-\varepsilon_0}+M_2+\varepsilon_0\bigg)|e_{-\gamma}(t,T)|,
\end{aligned}
$$

then

$$
\begin{aligned}
& \frac{1}{M_1+\varepsilon_0}|x_1(t)-x_2(t)|+|y_1(t)-y_2(t)| \\
& <2\bigg(\frac{M_1+\varepsilon_0}{m_1-\varepsilon_0}+M_2+\varepsilon_0\bigg)|e_{-\gamma}(t,T)|. \qquad (25)
\end{aligned}
$$

It follows from (25) that

$$\lim_{t\to+\infty}|x_1(t)-x_2(t)|=0,\ \lim_{t\to+\infty}|y_1(t)-y_2(t)|=0.$$

From the above discussion, we can see that system (1) is globally attractive. This completes the proof. □

Together with Theorem 15 and Lemma 17, we can obtain the following theorem.

Theorem 18. *Assume that the conditions* $(H_1)-(H_5)$ *hold, then system (1) with initial condition (2) has a unique globally attractive positive periodic solution in shifts* $\delta_\pm$.

5 Numerical examples

Example 1. Consider the following system on time scale $\mathbb{T} = \mathbb{R}$:

$$\begin{cases} x^\Delta(t) &= x(t)[0.8 + 0.2\sin\frac{\pi}{2}t \\ & \quad -(0.045 + 0.005\sin\frac{\pi}{2}t)x(t) \\ & \quad -x(\sigma(t)) - 0.2y(t)], \\ y^\Delta(t) &= -(0.4 + 0.1\cos\frac{\pi}{2}t)y(t) \\ & \quad +(0.015 + 0.005\sin\frac{\pi}{2}t)x(t). \end{cases} \tag{26}$$

Let $t_0 = 0$, then $\delta_+^\omega(t) = t+2$. By a direct calculation, we can get

$$\begin{aligned} & r^u = 1, r^l = 0.6, a^u = 0.05, a^l = 0.04, \\ & b^u = b^l = 1, c^u = c^l = 0.2, \eta^u = 0.5, \\ & \eta^l = 0.3, g^u = 0.02, g^l = 0.01, \\ & e_r(0, 2) = 0.2019, \\ & M_1 = 1,\ M_2 = 0.0667, \\ & a^u M_1 + c^u M_2 = 0.0633. \end{aligned}$$

Obviously, $e_r(t_0, \delta_+^\omega(t_0)) < 1$, $r^l > a^u M_1 + c^u M_2$, $a^l - g^u > 0$, $\eta^l - c^u > 0$, that is, the conditions $(H_1) - (H_5)$ hold. According to Theorem 18, system (26) has a unique globally attractive positive periodic solution in shifts $\delta_\pm$.

Example 2. Consider the following system on time scale $\mathbb{T} = 2^{\mathbb{N}_0}$:

$$\begin{cases} x^\Delta(t) &= x(t)\Big[\frac{1}{5t} - \frac{2}{t}x(t) - \frac{40}{t}x(\sigma(t)) \\ & \quad -\frac{1}{20t}y(t)\Big], \\ y^\Delta(t) &= -\frac{1}{3t}y(t) + \frac{1}{5t}x(t). \end{cases} \tag{27}$$

Let $t_0 = 1$, then $\delta_+^\omega(t) = 4t$. By a direct calculation, we can get

$$\begin{aligned} & r^u = 0.2, r^l = 0.05, a^u = 2, a^l = 0.5, \\ & b^u = 40, b^l = 10, \\ & c^u = 0.0500, c^l = 0.0125, \eta^u = 0.0125, \\ & \eta^l = 0.0833, g^u = 0.2, g^l = 0.05, \\ & e_r(1, 4) = 0.6818, \\ & M_1 = 0.02,\ M_2 = 0.0480, \\ & a^u M_1 + c^u M_2 = 0.0424. \end{aligned}$$

Obviously, $e_r(t_0, \delta_+^\omega(t_0)) < 1$, $r^l > a^u M_1 + c^u M_2$, $a^l - g^u > 0$, $\eta^l - c^u > 0$, that is, the conditions $(H_1) - (H_5)$ hold. According to Theorem 18, system (27) has a unique globally attractive positive periodic solution in shifts $\delta_\pm$.

6 Conclusion

This paper studied a nonlinear dynamic equation with feedback control on time scales. Based on the theory of calculus on time scales, by using a fixed point theorem of strict-set-contraction, some criteria are established for the existence of positive periodic solutions in shifts $\delta_\pm$ of the system; then, by using some differential inequalities and constructing a suitable Lyapunov functional, sufficient conditions which guarantee the global attractivity of the system are obtained.

The results obtained in this paper can be applied to systems on more general time scales, not only time scales are unbounded above and below. So the field of applications of one dynamic system is widened. Besides, the methods used in this paper can be applied to study many other dynamic systems.

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