Utility portfolio optimization with liability and multiple risky assets under the extended CIR model

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Abstract: This paper studies an asset and liability management problem with extended Cox-Ingersoll-Ross (CIR) interest rate, where the financial market is composed of one risk-free asset and multiple risky assets and one zero-coupon bond. We assume that risk-free interest rate is driven by extended CIR interest rate model, while liability is modeled by Brownian motion with drift and is generally correlated with stock price. Firstly, we use stochastic optimal control theory to obtain Hamilton-Jacobi-Bellman (HJB) equation for the value function and choose power utility and exponential utility for our analysis. Secondly, we obtain the closed-form solutions to the optimal investment strategies by applying variable change technique. Finally, a numerical example is presented to analyze the dynamic behavior of the optimal investment strategy and provide some economic implications for our results.

Key–Words: CIR interest rate model; liability process; dynamic portfolio selection; stochastic optimal control; optimal investment strategy;

1 Introduction

Dynamic portfolio selection problems with stochastic interest rates have been paid much more attentions to in recent years. The Vasicek model (1977) and Cox-Ingersoll-Ross(CIR) model (1985) were the most important models which described the term structure of interest rate dynamics. Stanton (1997) presented a nonparametric technique to estimate the drift and diffusion of the short rate, and the market price and the interest rate risk. Korn and Kraft (2001) used stochastic optimal control approach to study portfolio problems with stochastic interest rates and obtained the closed-form solutions of the optimal portfolios, and further presented verification theorem of the optimal solutions in the stochastic interest rate environments. Deelstra et al.(2000, 2003) investigated an investment and consumption problem in a CIR framework and the optimal policy for an insurer in the presence of a minimum guarantee respectively. Later, Grasselli (2003) studied an investment problem where the interest rate followed the CIR dynamics and obtained the optimal investment strategy in explicit form for HARA utility. Bielecki et al.(2005) focused on risk sensitive portfolio management problem with CIR interest rate dynamics. Liu (2007) assumed that risk-free interest rate and the appreciate rate and volatility term were functions of state variable, which followed Markovian diffusion process, and applied stochastic optimal control approach to study an investment and consumption problem in the finite horizon. Gao (2008) investigated a defined contribution pension funds problem with affine interest rate dynamics, which included the Vasicek model and the CIR model, and obtained the explicit solution of the optimal investment strategy. Li and Wu (2009) was concerned with a dynamic portfolio selection problem with CIR interest rate and Heson's stochastic volatility and obtained the explicit expression of the optimal investment policy. Ferland and Watier (2010) applied backward stochastic differential equation theory to study the mean-variance model with extended CIR interest rate dynamics and presented the explicit expressions of the optimal investment strategy and the efficient frontier.

Nowadays, the asset and liability management (ALM) problem is of both theoretical interest and practical importance, and has attracted more and more attentions in the last decade in the actuarial science and financial modelling. In recent years, many scholars have studied ALM problems in different situations. Leippold et al.(2004), Yi et al.(2008), Chen and Yang (2011) and Yao et al.(2013) studied mean-variance ALM problems in a multi-period environment under different market assumptions. Chiu and Li (2006), Xie et al.(2008) and Zeng and Li

(2011) investigated mean-variance ALM problem in a continuous-time framework under different liability processes. Chen et al.(2008), Xie (2009) and Li and Shu (2011) considered continuous-time mean-variance ALM problems in the regime switching setting; Chiu and Wong (2011, 2013) focused on dynamic portfolio selection problems with cointegrated assets both in a mean-variance framework and expected utility framework respectively. Later, Chiu and Wong (2012, 2013) investigated mean-variance ALM problems, where the prices of the assets are cointegrated. Leippold et al.(2011) and Yao et al.(2013) concerned mean-variance ALM problems with endogenous liability both in a multi-period setting and continuous-time setting respectively. In the above literature mentioned, risk-free interest rate was all kept fixed. Such an assumption is too restrictive for many financial institutions. In fact, the interest rate isn't always fixed in the real-world environments. Hence, considering ALM problems in the stochastic interest rate environments for many financial institutions or investor is more practice.

To the best of our knowledge, the ALM problem with CIR interest rate dynamics has not been reported in the existing literatures. In this paper, we assumed that risk-free interest rate is driven by the CIR model, and there are multiple risky assets and one zero-coupon bond in the financial market. The liability process is governed by Brownian motion with drift and is generally correlated with stock price dynamics. We firstly use dynamic programming principle to get the HJB equation for the value function and investigate the optimal investment strategies in the power utility and exponential utility cases. Due to the stochastic interest rate model and liability process, those factors make the HJB equation more sophisticated. Fortunately, we introduce a differential operator to transform the equation (16) into (23), which is easily solved directly. Secondly, we obtain the closed-form solutions to the optimal investment strategies by applying variable change technique. Finally, a numerical example is presented to analyze the dynamic behavior of the optimal investment strategy and provide some economic implications for our results. There are three main contribution in this paper: (i) the ALM problem with CIR interest rate dynamics is studied; (ii) the explicit expressions of the optimal investment strategies are obtained in the power utility and exponential utility cases; (iii) a verification theorem for the portfolio optimization problem with CIR interest rate dynamics is provided.

The rest of this paper is organized as follows. The problem formulation is presented in Section 2. In Section 3, we use dynamic programming principle to obtain the HJB equation and obtain the explicit expression and geometric structure of the optimal investment strategies in the power utility and exponential utility cases by applying variable change technique. Section 4 presents a numerical example to illustrate the effect of market parameters on the optimal investment strategies and gives some economic implications. Section 5 concludes the paper.

2 Problem formulation

In this section we provide a dynamic portfolio selection problem with liability process and extended CIR interest rate.

Throughout this paper, we assume that $(\cdot)'$ represents the transpose of a vector or a matrix, $[0, T]$ represents the fixed and finite investment horizon, $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{0\leqslant t\leqslant T}, P)$ represents a given filtered complete probability space, where $\{\mathcal{F}_t\}_{0\leqslant t\leqslant T}$ is a filtration and each $\mathcal{F}_t$ can be interpreted as the information available at time t, and any decision made at time t is based on those information.

2.1 Financial market

Assume that the financial market is composed of one risk-free asset and multiple risky assets and one zero-coupon bond.

One risk-free asset is interpreted as cash or bank account, whose price at time t is denoted by $S_0(t)$. Then $S_0(t)$ evolves according to

$$\frac{dS_0(t)}{S_0(t)} = r(t)dt, \quad S_0(0) = 1, \tag{1}$$

where $r(t)$ is risk-free interest rate.

In this paper, we suppose that the dynamics of short rate $r(t)$ is driven by CIR interest rate model:

$$dr(t) = (a - br(t))dt - \sigma_r\sqrt{r(t)}dW_r(t),$$
$$r(0) = r_0 > 0, \tag{2}$$

where $W_r(t)$ is a one-dimension standard Brownian motion defined on $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{0\leqslant t\leqslant T}, P)$, a, b and σ_r are all positive real constants and satisfy the condition: $2a > \sigma_r^2$. It leads to that $r(t) > 0$ for all $t \in [0, T]$.

Multiple risky assets is the stocks, whose price of the ith stock at time t is denoted by $S_i(t)$, $i = 1, 2, \cdots, n$. Then the dynamics of $S_i(t)$ can be described by (referring to Deelstra and Grasselli et al.(2003), Ferland and Watier (2010)):

$$\frac{dS_i(t)}{S_i(t)} = r(t)dt + \sum_{j=1}^{n}\sigma_{ij}(dW_j(t) + \lambda_j dt)+$$

$$\sigma_{ir}\sqrt{r(t)}(dW_r(t) + \lambda_r\sqrt{r(t)}dt), \quad S_i(0) = s_i > 0, \tag{3}$$

where $W_S(t) = (W_1(t), W_2(t), \cdots, W_n(t))'$ is a $n-$dimension independent and standard Brownian motion defined on $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{0\leqslant t\leqslant T}, P)$ and is independent of $W_r(t)$. In addition, $\Sigma_S = (\sigma_{ij})_{n\times n}$ represents the volatility matrix of the stock. Letting $\Sigma_r = (\sigma_{1r}, \sigma_{2r}, \cdots, \sigma_{nr})'$, $\Lambda = (\lambda_1, \lambda_2, \cdots, \lambda_n)'$, then $\Sigma_r\sqrt{r(t)}$ represents the volatility of stock price generated by the volatility of interest rate, $\lambda_r\sqrt{r(t)}$ and Λ can be taken as risk compensation coefficient vector generated by the volatility risk of the stock and the risk of interest rate respectively. It implies that the stock prices are influenced by the short rate and corresponding market price of risk.

The zero-coupon bond with maturity T, and the price of zero-coupon bond at time t is denoted by $B(t,T)$, then the dynamics of $B(t,T)$ can be represented by

$$\frac{dB(t,T)}{B(t,T)} = r(t)dt + \sigma_B(t)(dW_r(t) + \lambda_r\sqrt{r(t)}dt),$$

$$B(T,T) = 1, \tag{4}$$

where $\sigma_B(t) = \sigma_r(t)h(t)\sqrt{r(t)}$, and we have

$$h(t) = \frac{2(1 - e^{m(T-t)})}{m - (b - \lambda_r\sigma_r) + e^{m(T-t)}(m + b - \lambda_r\sigma_r)},$$

$$m = \sqrt{(b - \lambda_r\sigma_r)^2 + 2\sigma_r^2}. \tag{5}$$

2.2 Liability process

Assume that an investor is equipped with an initial endowment $w_0 > 0$ and an initial liability $l_0 > 0$ at time $t = 0$, then the net initial wealth of an investor is given by $x_0 = w_0 - l_0$. Suppose that the accumulative liability of the investor at time t is denoted by $L(t)$, then $L(t)$ can be described by the following Brownian motion with drift:

$$dL(t) = udt + vdW_L(t), \quad L(0) = l_0 > 0, \tag{6}$$

where u and v are the positive constants, and $W_L(t)$ is a one-dimension standard Brownian motion on $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{0\leqslant t\leqslant T}, P)$. In this paper, we assume that $W_L(t)$ is correlated with stock prices and the correlation coefficient between $W_L(t)$ and $W_i(t)$ is denoted by ρ_i, $i = 1, 2, \cdots, n$. Letting $\rho = (\rho_1, \rho_2, \cdots, \rho_n)'$, then $W_L(t)$ can be expressed as: $W_L(t) = \rho' W_S(t) + \sqrt{1 - \|\rho\|^2}\tilde{W}_L(t)$, where $\tilde{W}_L(t)$ is a one-dimension standard Brownian motion on $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{0\leqslant t\leqslant T}, P)$ and is independent of $W_S(t)$. Therefore, liability process $L(t)$ can be rewritten as

$$dL(t) = udt + v\rho' dW_S(t) + v\sqrt{1 - \|\rho\|^2}d\tilde{W}_L(t),$$

$$L(0) = l_0 > 0. \tag{7}$$

2.3 Wealth process

Assume that the amount invested in the ith stock at time t is denoted by $\pi_i(t)$, $i = 1, 2, \cdots, n$, and the amount invested in the zero-coupon bond is denoted by $\pi_B(t)$, then the amount invested in the risk-free asset is given by $\pi_0(t) = X(t) - \sum\limits_{i=1}^{n}\pi_i(t) - \pi_B(t)$, where $X(t)$ represents the net wealth of an investor at time t. Suppose that the financial market is frictionless and is allowed to short-selling the stock. Letting $\pi_S(t) = (\pi_1(t), \pi_2(t), \cdots, \pi_n(t))'$, then the net wealth process under trading strategy $\pi_S(t)$ and $\pi_B(t)$ can be written as:

$$dX(t) = (X(t) - \sum_{i=1}^{n}\pi_i(t) - \pi_B(t))\frac{dS_0(t)}{S_0(t)}$$

$$+\sum_{i=1}^{n}\pi_i(t)\frac{dS_i(t)}{S_i(t)} + \pi_B(t)\frac{dB(t,T)}{B(t,T)} - dL(t).$$

Taking (1), (3), (4) and (7) into consideration, we can get

$$\begin{aligned} dX(t) = &[r(t)X(t) + \pi_S(t)(\Sigma_S\Lambda + \Sigma_r\lambda_r r(t)) \\ &+\pi_B(t)\sigma_B(t)\lambda_r\sqrt{r(t)} - u]dt \\ &+(\pi'_S(t)\Sigma_S - v\rho')dW_S(t) - v\sqrt{1 - \|\rho\|^2}d\tilde{W}_L(t) \\ &+(\pi'_S(t)\Sigma_r\sqrt{r(t)} + \pi_B(t)\sigma_B(t))dW_r(t), \end{aligned} \tag{8}$$

with the initial value $X(0) = x_0$.

2.4 Optimization criterion

Definition 1(Admissible strategy). Trading strategy $\pi_S(t)$ and $\pi_B(t)$ are admissible if the following conditions are satisfied:

(i) $\pi_S(t)$ and $\pi_B(t)$ are progressively measurable;

(ii) $E\{\int_0^T [(\pi'_S(t)\Sigma_S - v\rho')^2 + (v\sqrt{1 - \|\rho\|^2})^2$

$+(\pi'_S(t)\Sigma_r\sqrt{r(t)} + \pi_B(t)\sigma_B(t))^2]dt\} < \infty$;

(iii) For all initial conditions (t_0, r_0, x_0), the wealth process $X(t)$ with $X(0) = x_0$ has a pathwise unique solution.

We denote the set of all admissible strategies by Γ, and the optimization problem of the investor in the utility framework can be expressed as:

$$\underset{\pi_S\in\Gamma,\pi_B\in\Gamma}{Maximize}\,\mathbb{E}U(X(T)). \tag{9}$$

where $U(\cdot)$ represents utility function and satisfies the condition: the first-order derivative $\dot{U}(x) > 0$ and the second-order derivative $\ddot{U}(x) < 0$.

3 The optimal portfolios

In this section, we apply dynamic programming principle and variable change technique to solve the problem (9) and choose power utility and exponential utility for our analysis.

We define the value function $J(t, r, x)$ of the problem (9) as

$$J(t,r,x) = \sup_{\pi_S\in\Gamma,\pi_B\in\Gamma} \mathbb{E}[U(X(T))\,|X(t) = x,\ r(t) = r]$$

with boundary condition: $J(T, r, x) = U(x)$.

Theorem 1.. Assume that $J(t, r, x)$ is continuously differentiable with respect to $t \in [0, T]$, and twice continuously differentiable with respective to $(r, x) \in \mathbb{R}\times\mathbb{R}$, then $J(t, r, x)$ satisfies the following HJB equation:

$$\begin{aligned}\sup_{\pi_S\in\Gamma,\pi_B\in\Gamma} \{J_t &+ [rx + \pi_S(t)(\Sigma_S\Lambda + \Sigma_r\lambda_r r)\\ &+\pi_B(t)\sigma_B(t)\lambda_r\sqrt{r} - u]J_x\\ &+\frac{1}{2}[(\pi'_S(t)\Sigma_S - v\rho')^2 + (v\sqrt{1-\|\rho\|^2})^2\\ &+(\pi'_S(t)\Sigma_r\sqrt{r} + \pi_B(t)\sigma_B(t))^2]J_{xx}\\ &+(a-br)J_r + \frac{1}{2}\sigma_r^2 rJ_{rr}\\ &-\sigma_r\sqrt{r}[\pi'_S(t)\Sigma_r\sqrt{r} + \pi_B(t)\sigma_B(t)]J_{xr}\} = 0. \end{aligned}\tag{10}$$

where $J_t, J_r, J_{rr}, J_x, J_{xx}, J_{xr}$ represent the first-order and second-order partial derivatives of the value function with respect to the variables t, r, x.

Proof. According to Theorem 3.1 of Lin and Li (2011), we assume that

$$J(t,r,x) = \sup_{\pi_S\in\Gamma,\pi_B\in\Gamma} \mathbb{E}[J(\tilde{\theta}, r(\tilde{\theta}), X^{\pi_S,\pi_B}(\tilde{\theta}))].$$

For any stopping time $\tilde{\theta} \in \Re_{t,T}$, where $\Re_{t,T}$ represents the set of all stopping time valued in $[t, T]$.

Considering the time $\tilde{\theta} = t+h$, for arbitrary $\pi_S \in \Gamma,\ \pi_B \in \Gamma$, we have

$$J(t,r,x) \geqslant \mathbb{E}[J(t+h, r(t+h), X^{\pi_S,\pi_B}(t+h))].$$

Applying $It\hat{o}'s$ formula between t and $t + h$, we have

$$J(t+h, r(t+h), X^{\pi_S,\pi_B}(t+h)) = J(t,r,x)$$

$+\int_t^{t+h}(\frac{\partial J}{\partial t} + \ell^{\pi_S,\pi_B}J)(u, r(u), X^{\pi_S,\pi_B}(u))du+$ local martingale.

where $\ell^{\pi_S,\pi_B}J$ is defined by

$$\begin{aligned}\ell^{\pi_S,\pi_B}J &= [rX_t + \pi_S(t)(\Sigma_S\Lambda + \Sigma_r\lambda_r r)\\ &+\pi_B(t)\sigma_B(t)\lambda_r\sqrt{r} - u]J_x\\ &+\frac{1}{2}[(\pi'_S(t)\Sigma_S - v\rho')^2 + (v\sqrt{1-\|\rho\|^2})^2\\ &+(\pi'_S(t)\Sigma_r\sqrt{r} + \pi_B(t)\sigma_B(t))^2]J_{xx}\\ &+(a-br)J_r + \frac{1}{2}\sigma_r^2 rJ_{rr}\\ &-\sigma_r\sqrt{r}[\pi'_S(t)\Sigma_r\sqrt{r} + \pi_B(t)\sigma_B(t)]J_{xr}.\end{aligned}$$

Then we obtain

$$\int_t^{t+h}(\frac{\partial J}{\partial t} + \ell^{\pi_S,\pi_B}J)(u, r(u), X^{\pi_S,\pi_B}(u))du \leqslant 0.$$

Dividing by h and letting $h \to 0$, this leads to

$$\frac{\partial J}{\partial t} + \ell^{\pi_S,\pi_B}J \leqslant 0.$$

On the other hand, suppose that $\pi_S^* \in \Gamma$ and $\pi_B^* \in \Gamma$ are the optimal investment strategies, then we have

$$J(t,r,x) = \mathbb{E}[J(t+h, r(t+h), X^{\pi_S^*,\pi_B^*}(t+h))].$$

By similar arguments as above, we obtain

$$\frac{\partial J}{\partial t} + \ell^{\pi_S^*,\pi_B^*}J = 0.$$

Summarizing the above arguments, we obtain that $J(t, r, x)$ should satisfy

$$\frac{\partial J}{\partial t} + \sup_{\pi_S\in\Gamma,\pi_B\in\Gamma}\ell^{\pi_S,\pi_B}J = 0.$$

This completes the proof of Theorem 1.□

The first-order maximizing conditions for the optimal strategy yield:

$$\pi_S(t) = -(\Sigma'_S)^{-1}\Lambda\frac{J_x}{J_{xx}} + (\Sigma'_S)^{-1}\rho v, \tag{11}$$

$$\begin{aligned}\pi_B(t) = &-\frac{\lambda_r\sqrt{r} - \Lambda'\Sigma_S^{-1}\Sigma_r\sqrt{r}}{\sigma_B(t)}\cdot\frac{J_x}{J_{xx}}\\ &+\frac{\sigma_r\sqrt{r}}{\sigma_B(t)}\cdot\frac{J_{rx}}{J_{xx}} - \frac{\rho' v\Sigma_S^{-1}\Sigma_r\sqrt{r}}{\sigma_B(t)}.\end{aligned}\tag{12}$$

Plugging (11) and (12) into (10), we get

$$J_t + rxJ_x + (\Lambda'\rho v - u)J_x$$
$$+\frac{1}{2}v^2(1-\|\rho\|^2)J_{xx} + (a-br)J_r$$
$$+\frac{1}{2}\sigma_r^2 rJ_{rr} - \frac{1}{2}(\|\Lambda\|^2 + \lambda_r^2 r)\frac{J_x^2}{J_{xx}}$$
$$+\lambda_r\sigma_r r\frac{J_xJ_{rx}}{J_{xx}} - \frac{1}{2}\sigma_r^2 r\frac{J_{rx}^2}{J_{xx}} = 0. \quad (13)$$

The following subsection, we try our best to solve the second-order nonlinear partial differential equation (13) by conjecturing the structure of the value function.

3.1 Power utility

Assume that power utility is expressed as

$$U(x) = \frac{x^\eta}{\eta}, \quad \eta < 1, \quad \eta \neq 0.$$

Conjecturing a solution to (13) with the following form:

$$J(t,r,x) = \frac{(x-g(t,r))^\eta}{\eta}f(t,r),$$
$$g(T,r) = 0, \quad f(T,r) = 1. \quad (14)$$

The partial derivatives of (14) are given by

$$J_t = -(x-g)^{\eta-1}g_tf + \frac{(x-g)^\eta}{\eta}f_t,$$
$$J_r = -(x-g)^{\eta-1}g_rf + \frac{(x-g)^\eta}{\eta}f_r,$$
$$J_x = (x-g)^{\eta-1}f,$$
$$J_{xx} = (\eta-1)(x-g)^{\eta-2}f,$$
$$J_{rx} = -(\eta-1)(x-g)^{\eta-2}g_rf + (x-g)^{\eta-1}f_r,$$
$$J_{rr} = (\eta-1)(x-g)^{\eta-2}g_r^2f + \frac{(x-g)^\eta}{\eta}f_{rr}$$
$$-2(x-g)^{\eta-1}f_rg_r - (x-g)^{\eta-1}g_{rr}f.$$

Putting the above partial derivatives in (13), we get

$$\frac{(x-g)^\eta}{\eta}\Big[f_t + (\eta r - \frac{\eta}{2(\eta-1)}(\|\Lambda\|^2 + \lambda_r^2 r))f$$
$$+(a - br + \frac{\eta}{\eta-1}\lambda_r\sigma_r r)f_r$$
$$+\frac{1}{2}\sigma_r^2 rf_{rr} - \frac{\eta}{2(\eta-1)}\sigma_r^2 r\frac{f_r^2}{f}\Big]$$
$$+\frac{1}{2}(\eta-1)(x-g)^{\eta-2}v^2(1-\|\rho\|^2)f$$
$$+(x-g)^{\eta-1}\big[g_t + (a - br + \lambda_r\sigma_r r)g_r$$
$$+\frac{1}{2}\sigma_r^2 rg_{rr} - rg - (\Lambda'\rho v - u)\big] = 0.$$

Eliminating the dependence on the variable x, we get the following two partial differential equations under the condition: $1 - \|\rho\|^2 = 0$.

$$f_t + (\eta r - \frac{\eta}{2(\eta-1)}(\|\Lambda\|^2 + \lambda_r^2 r))f$$
$$+(a - br + \frac{\eta}{\eta-1}\lambda_r\sigma_r r)f_r + \frac{1}{2}\sigma_r^2 rf_{rr}$$
$$-\frac{\eta}{2(\eta-1)}\sigma_r^2 r\frac{f_r^2}{f} = 0, \quad f(T,r) = 1; \quad (15)$$
$$g_t - rg + (a - br + \lambda_r\sigma_r r)g_r$$
$$+\frac{1}{2}\sigma_r^2 rg_{rr} + u - \Lambda'\rho v = 0, \quad g(T,r) = 0. \quad (16)$$

Lemma 1. Assume that the solution of (15) is of the structure $f(t,r) = e^{D_1(t)+D_2(t)r}$ with boundary conditions: $D_1(T) = 0$ and $D_2(T) = 0$, then under the condition: $\eta < \frac{b^2}{(\lambda_r\sigma_r - b)^2 + 2\sigma_r^2}$, $D_1(t)$ and $D_2(t)$ are given by (22) and (21), respectively.

Proof. Substituting $f(t,r) = e^{D_1(t)+D_2(t)r}$ into (15), we get

$$e^{D_1(t)+D_2(t)r}\Big\{\dot{D}_1(t) - \frac{\eta}{2(\eta-1)}\|\Lambda\|^2 + aD_2(t)$$
$$+r[\dot{D}_2(t) + (\eta - \frac{\eta}{2(\eta-1)}\lambda_r^2)$$
$$+(\frac{\eta}{\eta-1}\lambda_r\sigma_r - b)D_2(t) - \frac{1}{2(\eta-1)}\sigma_r^2 D_2^2(t)]\Big\} = 0.$$

Further, we can get the following two equations:

$$\dot{D}_2(t) + (\eta - \frac{\eta}{2(\eta-1)}\lambda_r^2) + (\frac{\eta}{\eta-1}\lambda_r\sigma_r - b)D_2(t)$$
$$-\frac{1}{2(\eta-1)}\sigma_r^2 D_2^2(t) = 0, \quad D_2(T) = 0; \quad (17)$$
$$\dot{D}_1(t) - \frac{\eta}{2(\eta-1)}\|\Lambda\|^2 + aD_2(t) = 0, \quad D_1(T) = 0. \quad (18)$$

The equation (17) can be rewritten as

$$\frac{1}{m_1 - m_2}\int_t^T (\frac{1}{D_2(t) - m_1} - \frac{1}{D_2(t) - m_2})dD_2(t)$$
$$= \frac{\sigma_r^2}{2(\eta-1)}(T-t). \quad (19)$$

where m_1 and m_2 are two different roots of the following quadratic equation:

$$\frac{1}{2(\eta-1)}\sigma_r^2 D_2^2(t)-(\frac{\eta}{\eta-1}\lambda_r\sigma_r-b)D_2(t)$$
$$-(\eta-\frac{\eta}{2(\eta-1)}\lambda_r^2)=0.$$

i.e. under the condition $\eta<\frac{b^2}{(\lambda_r\sigma_r-b)^2+2\sigma_r^2}$, we get

$$m_{1,2}=\frac{\eta\lambda_r\sigma_r-(\eta-1)b}{\sigma_r^2}$$
$$\pm\frac{\sqrt{\eta(\eta-1)(\lambda_r\sigma_r-b)^2+(\eta-1)(2\eta\sigma_r^2-b^2)}}{\sigma_r^2}. \tag{20}$$

Solving the equation (19), we obtain

$$D_2(t)=\frac{m_1m_2(1-\exp\{\frac{\sigma_r^2}{2(\eta-1)}(m_1-m_2)(T-t)\})}{m_1-m_2\cdot\exp\{\frac{\sigma_r^2}{2(\eta-1)}(m_1-m_2)(T-t)\}}. \tag{21}$$

Integrating the both sides of (18), we get

$$D_1(t)=a\int_t^T D_2(t)dt-\frac{\eta}{2(\eta-1)}\left\|\Lambda\right\|^2(T-t). \tag{22}$$

Therefore, the proof of Lemma 1 is completed. □

Lemma 2. Suppose that the solution to (15) is of the structure $g(t,r)=(u-\Lambda'\rho v)\int_t^T\hat{g}(u,r)du$, then $\hat{g}(t,r)$ satisfies the following equation:

$$\hat{g}_t-r\hat{g}+(a-br+\lambda_r\sigma_r r)\hat{g}_r$$
$$+\frac{1}{2}\sigma_r^2 r\hat{g}_{rr}=0,\quad \hat{g}(T,r)=1. \tag{23}$$

Proof. We introduce a differential operator as follows.

$$\nabla g=-rg+(a-br+\lambda_r\sigma_r r)g_r+\frac{1}{2}\sigma_r^2 rg_{rr}.$$

Then (16) is rewritten as

$$g_t+\nabla g+u-\Lambda'\rho v=0,\quad g(T,r)=0. \tag{24}$$

Considering $g(t,r)=(u-\Lambda'\rho v)\int_t^T\hat{g}(u,r)du$, we yield

$$g_t=-(u-\Lambda'\rho v)\hat{g}(t,r)$$
$$=(u-\Lambda'\rho v)(\int_t^T\frac{\partial\hat{g}(u,r)}{\partial u}du-\hat{g}(T,r)),$$
$$g_r=(u-\Lambda'\rho v)\int_t^T\frac{\partial\hat{g}(u,r)}{\partial r}du,$$
$$g_{rr}=(u-\Lambda'\rho v)\int_t^T\frac{\partial^2\hat{g}(u,r)}{\partial r^2}du.$$

Further, we have

$$\nabla g=(u-\Lambda'\rho v)\int_t^T\nabla\hat{g}(u,r)du.$$

Inserting g_t and ∇g into

$$(u-\Lambda'\rho v)\int_t^T[\frac{\partial\hat{g}(u,r)}{\partial u}+\nabla\hat{g}(u,r)]du$$
$$-(u-\Lambda'\rho v)(\hat{g}(T,r)-1)=0.$$

In fact, we have

$$\frac{\partial\hat{g}(u,r)}{\partial u}+\nabla\hat{g}(u,r)=0,\quad \hat{g}(T,r)=1.$$

The proof of Lemma 2 is completed. □

Lemma 3. Letting $\hat{g}(t,r)=e^{D_3(t)+D_4(t)r}$ is the solution to the equation (23), where boundary conditions are given by $D_3(T)=0$ and $D_4(T)=0$, then $D_3(t)$ and $D_4(t)$ are determined by (28) and (27), respectively.

Proof. Putting $\hat{g}(t,r)=e^{D_3(t)+D_4(t)r}$ in the equation (23) yields:

$$e^{D_3(t)+D_4(t)r}\{\dot{D}_3(t)+aD_4(t)$$
$$+[\dot{D}_4(t)-1+(\lambda_r\sigma_r-b)D_4(t)+\frac{1}{2}\sigma_r^2D_4^2(t)]r\}=0.$$

We can decompose the above equation into the following two equations in order to eliminate the dependence on r:

$$\dot{D}_4(t)-1+(\lambda_r\sigma_r-b)D_4(t)$$
$$+\frac{1}{2}\sigma_r^2D_4^2(t)=0,\quad D_4(t)=0. \tag{25}$$

$$\dot{D}_3(t)+aD_4(t)=0,\quad D_3(t)=0. \tag{26}$$

Using the same analysis as the equations (17) and (18), we derive

$$D_4(t)=\frac{m_3m_4(1-\exp\{-\frac{1}{2}\sigma_r^2(m_3-m_4)(T-t)\})}{m_3-m_4\cdot\exp\{-\frac{1}{2}\sigma_r^2(m_3-m_4)(T-t)\}}. \tag{27}$$

$$D_3(t)=a\int_t^T D_4(t)dt. \tag{28}$$

where m_3 and m_4 are given by

$$m_{3,4}=\frac{b-\lambda_r\sigma_r}{\sigma_r^2}\pm\frac{\sqrt{(b-\lambda_r\sigma_r)^2+2\sigma_r^2}}{\sigma_r^2}.$$

Therefore, we complete the proof of Lemma 3. □

Further, we have

$$\frac{J_x}{J_{xx}}=\frac{1}{\eta-1}(x-g),$$

$$\frac{J_{rx}}{J_{xx}} = -g_r + \frac{1}{\eta-1}(x-g)\frac{f_r}{f}$$
$$= -g_r + \frac{1}{\eta-1}(x-g)D_2(t).$$

Finally, we obtain the optimal investment strategies in the power utility case.

Proposition 1. Assume that utility function is given by $U(x) = \frac{x^\eta}{\eta}$, with $\eta < 1$ and $\eta \neq 0$, then under the conditions: $\eta < Min\{\frac{b^2}{(\lambda_r\sigma_r - b)^2 + 2\sigma_r^2},\ 1\}$, $\eta \neq 0$ and $\|\rho\|^2 = 1$, the optimal investment strategies of the problem (9) are given by

$$\pi_S^*(t) = \frac{1}{1-\eta}(\Sigma_S')^{-1}\Lambda(X(t)-g) + (\Sigma_S')^{-1}\rho v, \tag{29}$$

$$\pi_B^*(t) = \frac{1}{1-\eta}\cdot\frac{\lambda_r\sqrt{r(t)} - \Lambda'\Sigma_S^{-1}\Sigma_r\sqrt{r(t)}}{\sigma_B(t)}(X(t)-g) +$$
$$\frac{\sigma_r\sqrt{r(t)}}{\sigma_B(t)}\cdot\left(\frac{1}{\eta-1}(X(t)-g)D_2(t) - g_r\right) - \frac{\rho' v\Sigma_S^{-1}\Sigma_r\sqrt{r(t)}}{\sigma_B(t)}. \tag{30}$$

where $g = g(t,r) = (u - \Lambda'\rho v)\int_t^T \hat{g}(u,r)du$, $\hat{g}(t,r) = e^{D_3(t)+D_4(t)r}$, $D_2(t)$, $D_3(t)$ and $D_4(t)$ are given by (21), (28) and (27), respectively.

Remark1. If the liability is not considered, i.e. $u = v = 0$, then $g = g(t,r) = 0$. Therefore, the optimal policy of the problem (9) for power utility is reduced to

$$\pi_S^*(t) = \frac{1}{1-\eta}(\Sigma_S')^{-1}\Lambda X(t),$$

$$\pi_B^*(t) = \frac{1}{1-\eta}\cdot\frac{\lambda_r\sqrt{r(t)} - \Lambda'\Sigma_S^{-1}\Sigma_r\sqrt{r(t)}}{\sigma_B(t)}X(t) +$$
$$\frac{1}{\eta-1}\cdot\frac{\sigma_r\sqrt{r(t)}}{\sigma_B(t)}X(t)D_2(t).$$

where $D_2(t)$ is given by(21).

Remark2. If $\eta \to 0$, then we have $D_2(t) = 0$. It is all well-known that power utility is degenerated to logarithm utility when $\eta \to 0$. Hence, the optimal policy of the problem (9) is given by

$$\pi_S^*(t) = (\Sigma_S')^{-1}\Lambda(X(t)-g) + (\Sigma_S')^{-1}\rho v.$$

$$\pi_B^*(t) = \frac{\lambda_r\sqrt{r(t)} - \Lambda'\Sigma_S^{-1}\Sigma_r\sqrt{r(t)}}{\sigma_B(t)}(X(t)-g) +$$
$$\frac{\sigma_r\sqrt{r(t)}}{\sigma_B(t)}\cdot(-g_r) - \frac{\rho' v\Sigma_S^{-1}\Sigma_r\sqrt{r(t)}}{\sigma_B(t)}.$$

where $g = g(t,r) = (u - \Lambda'\rho v)\int_t^T \hat{g}(u,r)du$, $\hat{g}(t,r) = e^{D_3(t)+D_4(t)r}$, $D_3(t)$ and $D_4(t)$ are given by (28) and (27), respectively.

3.2 Exponential utility

Assume that exponential utility is expressed as

$$U(x) = -\frac{1}{\beta}e^{-\beta x}, \quad \beta > 0.$$

Further, we suppose that the solution to (13) is of the following structure

$$J(t,r,x) = -\frac{1}{\beta}\exp\{-\beta K(t,r)(x - L(t,r)) + M(t,r)\}. \tag{31}$$

with boundary conditions: $K(T,r) = 1$, $L(T,r) = 0$, $M(T,r) = 0$.

Then we have

$$J_t = J(-\beta x K_t + \beta K_t L + \beta K L_t + M_t),$$
$$J_x = J(-\beta K), \quad J_{xx} = J(-\beta K)^2,$$
$$J_r = J(-\beta x K_r + \beta K_r L + \beta K L_r + M_r),$$
$$J_{rx} = J(-\beta K)(-\beta x K_r + \beta K_r L + \beta K L_r + M_r + \frac{K_r}{K}),$$
$$J_{rr} = J(-\beta x K_r + \beta K_r L + \beta K L_r + M_r)^2$$
$$+J(-\beta x K_{rr} + \beta K_{rr} L + 2\beta K_r L_r + \beta K L_{rr} + M_{rr}). \tag{32}$$

Putting (32) into (13), after some calculations, we obtain

$$J\{-\beta x[K_t + rK + (a - br + \lambda_r\sigma_r r)K_r$$
$$+\frac{1}{2}\sigma_r^2 r K_{rr} - \sigma_r^2 r\frac{K_r^2}{K}]$$
$$+\beta K\Big[\frac{L}{K}(K_t + (a - br + \lambda_r\sigma_r r)K_r + \frac{1}{2}\sigma_r^2 r K_{rr}$$
$$-\sigma_r^2 r\frac{K_r^2}{K}) + L_t + (a - br + \lambda_r\sigma_r r)L_r$$
$$+\frac{1}{2}\sigma_r^2 r L_{rr} + u - \Lambda'\rho v + \frac{1}{2}v^2(1 - \|\rho\|^2)\beta K\Big]$$
$$+M_t + (a - br + \lambda_r\sigma_r r - \sigma_r^2 r\frac{K_r}{K})M_r + \frac{1}{2}\sigma_r^2 r M_{rr}$$
$$-\frac{1}{2}(\|\Lambda\|^2 + \lambda_r^2 r + \sigma_r^2 r(\frac{K_r}{K})^2 - 2\lambda_r\sigma_r r\frac{K_r}{K})\Big\} = 0. \tag{33}$$

Eliminating the dependence on the variable x, we can decompose (33) into the following three partial differential equations:

$$K_t + rK + (a - br + \lambda_r\sigma_r r)K_r$$
$$+\frac{1}{2}\sigma_r^2 r K_{rr} - \sigma_r^2 r\frac{K_r^2}{K} = 0, \quad K(T,r) = 1; \tag{34}$$

$$\beta K\left[\frac{L}{K}(K_t+(a-br+\lambda_r\sigma_r r)K_r\right.$$
$$+\frac{1}{2}\sigma_r^2 rK_{rr}-\sigma_r^2 r\frac{K_r^2}{K})+L_t+u-\Lambda'\rho v$$
$$+\frac{1}{2}\sigma_r^2 rL_{rr}+(a-br+\lambda_r\sigma_r r)L_r$$
$$\left.+\frac{1}{2}v^2(1-\|\rho\|^2)\beta K\right]=0,\quad L(T,r)=0; \quad (35)$$

$$M_t+(a-br+\lambda_r\sigma_r r-\sigma_r^2 r\frac{K_r}{K})M_r+\frac{1}{2}\sigma_r^2 rM_{rr}$$
$$-\frac{1}{2}(\|\Lambda\|^2+\lambda_r^2 r+\sigma_r^2 r(\frac{K_r}{K})^2-2\lambda_r\sigma_r r\frac{K_r}{K})=0. \quad (36)$$

Lemma 4. Assume that the solution to (34) is of the form: $K(t,r)=e^{D_5(t)+D_6(t)r}$, with boundary conditions given by $D_5(T)=0$ and $D_6(T)=0$, then $D_6(t)$ and $D_5(t)$ are determined by (39) and (40), respectively.

Proof. Plugging $K(t,r)=e^{D_5(t)+D_6(t)r}$ into (34) yields

$$e^{D_5(t)+D_6(t)r}\{\dot{D}_5(t)+aD_6(t)$$
$$+[\dot{D}_6(t)+1+(\lambda_r\sigma_r-b)D_6(t)-\frac{1}{2}\sigma_r^2 D_6^2(t)]r\}=0.$$

Then we have

$$\dot{D}_6(t)+1+(\lambda_r\sigma_r-b)D_6(t)-\frac{1}{2}\sigma_r^2 D_6^2(t)=0, \quad (37)$$

$$\dot{D}_5(t)+aD_6(t)=0. \quad (38)$$

Solving (37) and (38), we get

$$D_6(t)=\frac{m_5m_6(1-\exp\{\frac{1}{2}\sigma_r^2(m_5-m_6)(T-t)\})}{m_5-m_6\cdot\exp\{\frac{1}{2}\sigma_r^2(m_5-m_6)(T-t)\}}, \quad (39)$$

$$D_5(t)=a\int_t^T D_6(t)dt. \quad (40)$$

where m_5 and m_6 are given by

$$m_{5,6}=\frac{\lambda_r\sigma_r-b}{\sigma_r^2}\pm\frac{\sqrt{(b-\lambda_r\sigma_r)^2+2\sigma_r^2}}{\sigma_r^2}.$$

The proof is completed. □

Lemma 5. Suppose that the solution to (35) is $L(t,r)=(u-\Lambda'\rho v)\int_t^T\hat{L}(u,r)du$ and $\hat{L}(t,r)=e^{D_7(t)+D_8(t)r}$, with boundary conditions: $D_7(T)=0$ and $D_8(T)=0$, then under the condition $\|\rho\|^2=1$, we obtain $D_8(t)=D_4(t)$ and $D_7(t)=D_3(t)$.

Proof. Taking (34) and the condition $\|\rho\|^2=1$ into considerations, we get

$$L_t-rL+(a-br+\lambda_r\sigma_r r)L_r$$
$$+\frac{1}{2}\sigma_r^2 rL_{rr}+u-\Lambda'\rho v=0,\quad L(T,r)=0. \quad (41)$$

Applying the same approach as (16) and referring to the proof of Lemma 2 and Lemma 3, we can easily get the results of Lemma 5. □

Lemma 6. Assume that the solution to (36) is expressed as $M(t,r)=D_9(t)+D_{10}(t)r$, with boundary conditions given by $D_9(T)=0$ and $D_{10}(T)=0$, then $D_{10}(t)$ and $D_9(t)$ are determined by (45) and (46), respectively.

Proof. Introducing $M(t,r)=D_9(t)+D_{10}(t)r$ in the equation (36), we derive

$$\dot{D}_9(t)+aD_{10}(t)-\frac{1}{2}\|\Lambda\|^2$$
$$+r[\dot{D}_{10}(t)+(\lambda_r\sigma_r-b-\sigma_r^2D_6(t))D_{10}(t)$$
$$-\frac{1}{2}(\lambda_r-\sigma_rD_6(t))^2]=0. \quad (42)$$

Comparing the coefficients on the both sides of (42), we get

$$\dot{D}_{10}(t)+(\lambda_r\sigma_r-b-\sigma_r^2D_6(t))D_{10}(t)$$
$$-\frac{1}{2}(\lambda_r-\sigma_rD_6(t))^2=0,\quad D_{10}(T)=0; \quad (43)$$

$$\dot{D}_9(t)+aD_{10}(t)-\frac{1}{2}\|\Lambda\|^2=0,\quad D_9(T)=0. \quad (44)$$

Solving (43) and (44) yields:

$$D_{10}(t)=-\frac{1}{2}e^{-\int_0^t(\lambda_r\sigma_r-b-\sigma_r^2D_6(t))dt}$$
$$\cdot\int_t^T(\lambda_r-\sigma_rD_6(t))^2e^{\int_0^t(\lambda_r\sigma_r-b-\sigma_r^2D_6(t))dt}dt, \quad (45)$$

$$D_9(t)=a\int_t^T D_{10}(t)dt-\frac{1}{2}\|\Lambda\|^2(T-t). \quad (46)$$

The proof of Lemma 6 is completed. □

Further, applying (32) and the results of Lemma 4, Lemma 5 and Lemma 6, we have

$$\frac{J_x}{J_{xx}}=-\frac{1}{\beta K},$$

$$\frac{J_{rx}}{J_{xx}}=\frac{K_r}{K}x-\frac{K_r}{K}L-L_r-\frac{M_r}{\beta K}-\frac{K_r}{\beta K^2}$$
$$=D_6(t)(x-L)-L_r-\frac{1}{\beta K}(D_{10}(t)+D_6(t)).$$

Finally, we can summarize the optimal investment strategies of the problem (9) for exponential utility maximization in the following Proposition 2.

Proposition 2. If utility function is given by $U(x) = -\frac{1}{\beta}e^{-\beta x}$, $\beta > 0$, then under the condition $\|\rho\|^2 = 1$, the optimal policies of the problem (9) are

$$\pi_S^*(t) = \frac{1}{\beta K}(\Sigma_S')^{-1}\Lambda + (\Sigma_S')^{-1}\rho v, \quad (47)$$

$$\pi_B^*(t) = \frac{1}{\beta K}\cdot\frac{\lambda_r\sqrt{r(t)} - \Lambda'\Sigma_S^{-1}\Sigma_r\sqrt{r(t)}}{\sigma_B(t)} + \frac{\sigma_r\sqrt{r(t)}}{\sigma_B(t)}\cdot[D_6(t)(X(t)-L) - L_r - \frac{1}{\beta K}(D_{10}(t)+D_6(t))] - \frac{\rho' v\Sigma_S^{-1}\Sigma_r\sqrt{r(t)}}{\sigma_B(t)}. \quad (48)$$

where $K = K(t,r) = e^{D_5(t)+D_6(t)r}$, $L = L(t,r) = (u - \Lambda'\rho v)\int_t^T \hat{L}(u,r)du$, $\hat{L}(t,r) = e^{D_7(t)+D_8(t)r}$, $D_5(t)$, $D_6(t)$, $D_7(t)$, $D_8(t)$ and $D_{10}(t)$ are given by Lemma 4-Lemma 6.

Remark3. If there is no liability, i.e. $u = v = 0$, and it leads to $L = L(t,r) = 0$. Therefore, the optimal policies with CIR interest rate dynamics for exponential utility ate given by

$$\pi_S^*(t) = \frac{1}{\beta K}(\Sigma_S')^{-1}\Lambda,$$

$$\pi_B^*(t) = \frac{1}{\beta K}\cdot\frac{\lambda_r\sqrt{r(t)} - \Lambda'\Sigma_S^{-1}\Sigma_r\sqrt{r(t)}}{\sigma_B(t)} + \frac{\sigma_r\sqrt{r(t)}}{\sigma_B(t)}\cdot[D_6(t)X(t) - \frac{1}{\beta K}(D_{10}(t)+D_6(t))].$$

where $K = K(t,r) = e^{D_5(t)+D_6(t)r}$, in addition, $D_5(t)$, $D_6(t)$ and $D_{10}(t)$ are given by (40), (39), (45), respectively.

4 Numerical analysis

This section provides a numerical example to illustrate the impact of market parameters on the optimal investment strategy. Assume that the financial market is composed of one risk-free asset and two risky assets and one zero-coupon bond. Throughout this section, unless otherwise stated, the basic parameters are given by $a = 0.35$, $b = 0.4$, $\sigma_r = 0.1$, $r(0) = 0.05$, $\Lambda = (0.4,\ 0.6)'$, $\Sigma_r = (0.16,\ 0.32)'$, $\lambda_r = 0.16$, $u = 200$, $v = 300$, $\rho = (0.6,\ 0.8)'$, $t = 0$, $T = 2$, $X(0) = 1000$, $\Sigma_S = \begin{pmatrix} 1.68 & 0.68 \\ 0.68 & 1.46 \end{pmatrix}$. Notice that in the following figures, the amount invested in the bank account is denoted by the thick line, i.e. $\pi_0^*(t)$, and the sum of the amount invested in the stocks is given by the dashed line, i.e. $\sum_{i=1}^2 \pi_i^*(t)$, and the amount invested in the zero-coupon bond is represented by the orange line, i.e. $\pi_B^*(t)$.

4.1 Sensitivity analysis in the power utility case

Under power utility, assume that the risky aversion factor is given by $\eta = -0.2$. It can be seen from every gragh of Fig.1-Fig.6 that how market parameters affect the optimal investment strategy. Some results are summarized as follows.

(a1) $\sum_{i=1}^2 \pi_i^*(t)$ is not sensitive to the parameter b, and $\pi_B^*(t)$ increases with respect to (w.r.t) the value of b, and $\pi_0^*(t)$ decreases w.r.t the parameter b. It is shown from the equation (1) that the expected value of interest rate will decrease with the increasing value of b. It means that the the bigger the value of b becomes, the smaller amount an investor invests the risk-free asset and the bigger amount in the zero-coupon bond.

(a2) $\sum_{i=1}^2 \pi_i^*(t)$ is not sensitive to the parameter σ_r as well, and $\pi_B^*(t)$ decreases in the value of σ_r, and $\pi_0^*(t)$ is increasing with the parameter σ_r. Notice that when the value of σ_r is increasing, the volatility of interest rate is increasing as well. So, it tells us that an investor should invest the less amount in the zero-coupon bond and invest the more amount in the risk-free asset.

(a3) $\sum_{i=1}^2 \pi_i^*(t)$ and $\pi_B^*(t)$ are all decreasing with the increasing value of the parameter u, and $\pi_0^*(t)$ increases w.r.t the parameter u. Noting that the bigger the value of u is, the bigger the expected value of liability is. It displays that an investor should invest the less amount of wealth in the risky assets and invest the more amount in the risk-free asset.

(a4) $\sum_{i=1}^2 \pi_i^*(t)$ and $\pi_B^*(t)$ increases w.r.t the parameter v, and $\pi_0^*(t)$ decreases w.r.t the paremeter v. In fact, the parameter v represents the volatility of liability. Hence, the larger the value of v is, the more risk the liability results into. It illustates that an investor should invest the more amount of wealth in the risky assets and invests the less amount in the risk-free asset in order to hedge the risk of liability.

(a5) $\sum_{i=1}^2 \pi_i^*(t)$ keeps fixed almost with the investment horizon T, while $\pi_B^*(t)$ decreases w.r.t the parameter T and $\pi_0^*(t)$ increases w.r.t the value of T. It shows that $\pi_B^*(t)$ is decreasing and $\pi_0^*(t)$ is increasing when the value of T is increasing. In fact, the bigger the value of T is, the bigger the value of $T - t$ is. Hence, it indicates that as time t elapses, an investor should invest the more amount of wealth in the zero-coupon bond and keeps the less amount in the risk-free asset.

(a6) $\sum_{i=1}^2 \pi_i^*(t)$ and $\pi_B^*(t)$ increases w.r.t risk aversion factor η and $\pi_0^*(t)$ decreases w.r.t the value of η. Under power utility, the risk aversion coefficient of an investor is denoted by $1-\eta$. Therefore, when the value of η is increasing, the risk aversion agree of investor is decreasing. It leads to that an investor would invest the more amount of wealth in the risky assets and invest the less amount in the risk-free asset.

4.2 Sensitivity analysis in the exponential utility case

Under exponential utility, suppose that $\beta = 0.001$ and $\lambda_r = 0.01$, the other market parameters keep fixed. Fig.7-Fig.12 illustrate that how market parameters impact on the optimal investment strategy. In addition, the following conclusions are drawn.

(b1) $\sum_{i=1}^{2} \pi_i^*(t)$ is not sensitive to the parameter b, while $\pi_B^*(t)$ increases with respect to b and $\pi_0^*(t)$ decreases with respect to b. This is to say, the bigger the value of b is, the more amount of wealth an investor invests in the zero-coupon bond and the less amount in the risk-free asset.

(b2) $\sum_{i=1}^{2} \pi_i^*(t)$ keeps fixed almost w.r.t the parameter σ_r, in the meantime, $\pi_B^*(t)$ decreases w.r.t the parameter b and $\pi_0^*(t)$ increases w.r.t the value of b. It implies that as the value of b becomes larger, an investor should invest the less amount of wealth in the zero-coupon bond and invest the more amount in the risk-free asset.

(b3) $\sum_{i=1}^{2} \pi_i^*(t)$ is fixed in the value of λ_r, while $\pi_B^*(t)$ is decreasing with the value of λ_r and $\pi_0^*(t)$ is increasing. It indicates that when the value of λ_r become larger, an investor should invest the less amount of wealth in the zero-coupon bond and invest the more amount in the risk-free asset.

(b4) $\sum_{i=1}^{2} \pi_i^*(t)$ and $\pi_B^*(t)$ increases with respect to the parameter v, while $\pi_0^*(t)$ decreases w.r.t the value of v. It tells us that when the volatility of liability is increasing, the investor should invest the more amount of wealth in the risky assets in order to hedge the risk resulted from the liability.

(b5) $\sum_{i=1}^{2} \pi_i^*(t)$ and $\pi_B^*(t)$ is decreasing in the value of T, while $\pi_0^*(t)$ increases w.r.t the parameter T. It implies that as time elapses, an investor would take more risk and invest more money in the risky assets.

(b6) $\sum_{i=1}^{2} \pi_i^*(t)$ and $\pi_B^*(t)$ decreases with the value of β, while $\pi_0^*(t)$ increases w.r.t the parameter β. It is well-known that the risk aversion coefficient in the exponential utility case is given by β. Therefore, the bigger the value of β is, the more risk aversion an investor is faced with. This is the reason why an investor would invest less money in the risky assets and invest more money in the risk-free asset.

5 Conclusions

In this paper, we have studied an asset and liability management problem with CIR interest rate dynam-

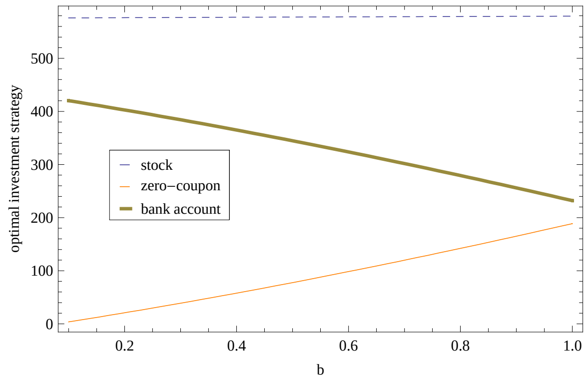

Figure 1: The impact of b on the optimal policies with power preference

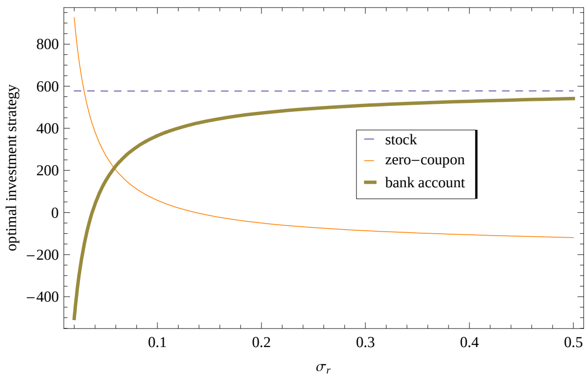

Figure 2: The impact of σ_r on the optimal policies with power preference

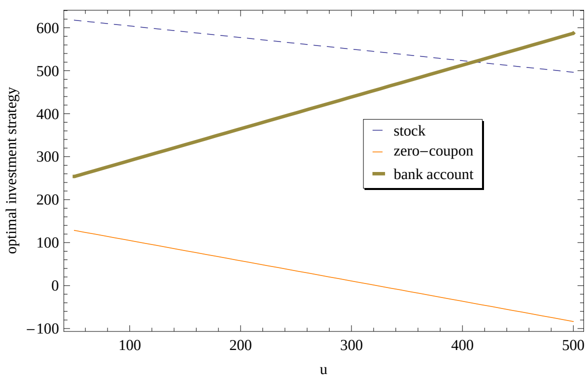

Figure 3: The impact of u on the optimal policies with power preference

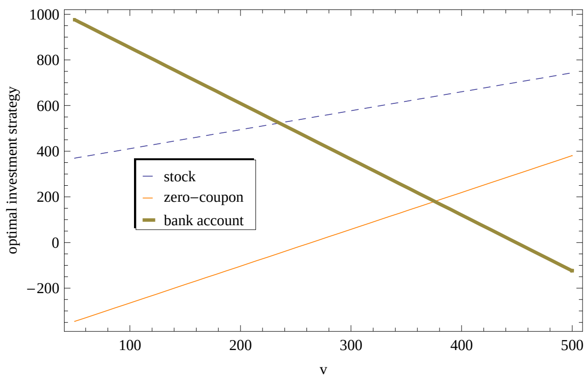

Figure 4: The impact of v on the optimal policies with power preference

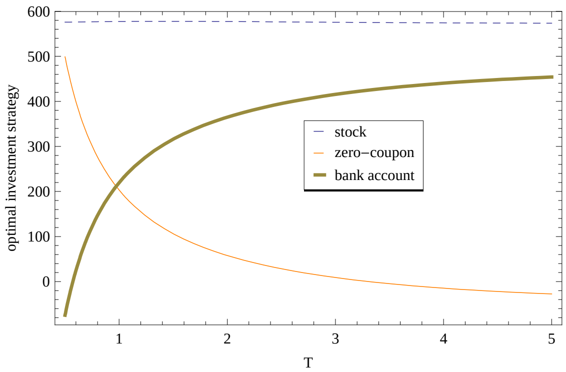

Figure 5: The impact of T on the optimal policies with power preference

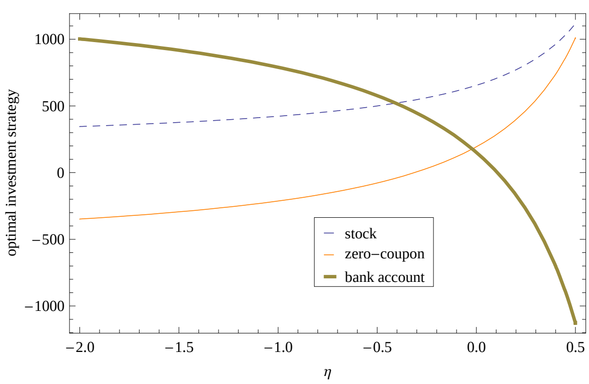

Figure 6: The impact of η on the optimal policies with power preference

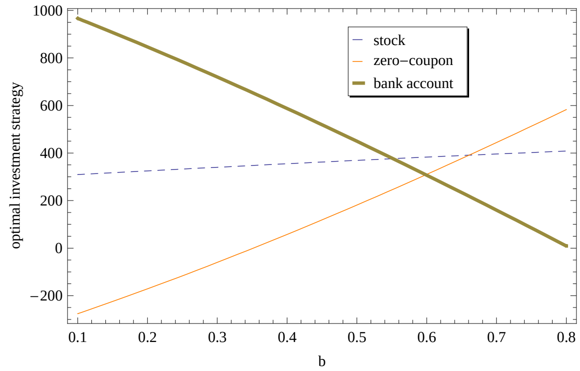

Figure 7: The impact of b on the optimal policies with exponential preference

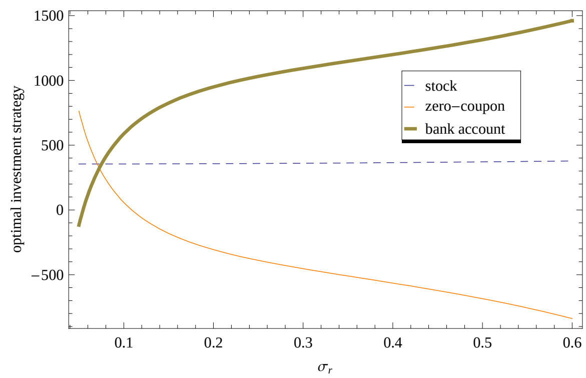

Figure 8: The impact of σ_r on the optimal policies with exponential preference

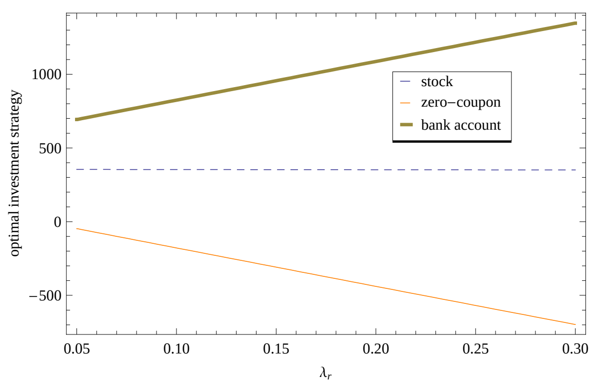

Figure 9: The impact of λ_r on the optimal policies with exponential preference

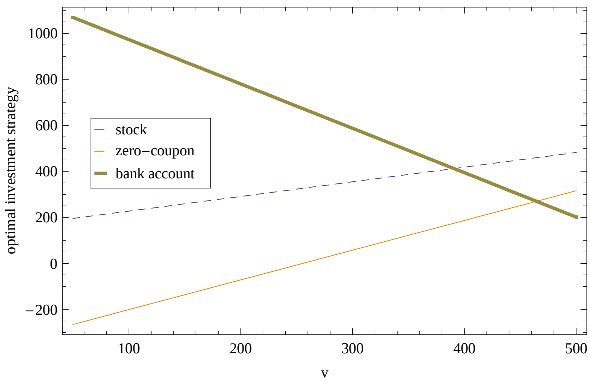

Figure 10: The impact of v on the optimal policies with exponential preference

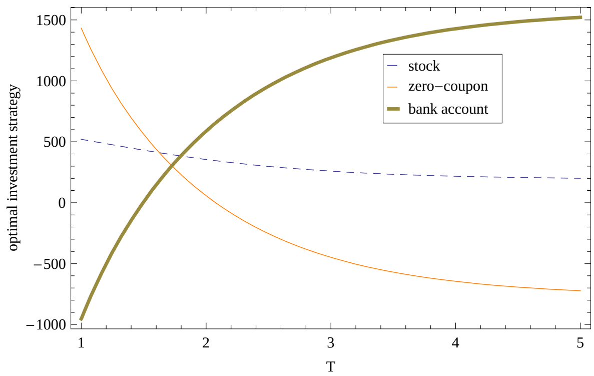

Figure 11: The impact of T on the optimal policies with exponential preference

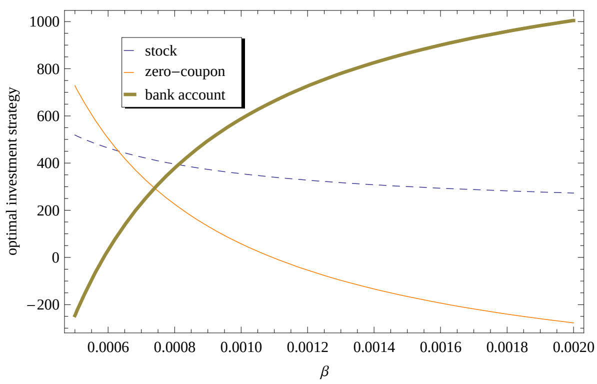

Figure 12: The impact of β on the optimal policies with exponential preference

ics. The financial market is composed of one risk-free asset and multiple risky assets and one zero-coupon bond. The liability process is assumed to be driven by Brownian motion with drift and be generally correlated with stock price dynamics, while the price processes of stocks and zero-coupon are affected by interest rate dynamics. The closed-form solutions to the optimal investment strategies for power utility and exponential utility are obtained. We also present a numerical example to analyze the influence of market parameters on the optimal investment strategies and provide some economic implications. Some important results are found: (i) the optimal investment strategy with liability and stochastic interest rate is more sophisticated than that with stochastic interest rate only; (ii) the optimal policies for power and exponential utility have opposite trend with respect to risk aversion factor; (iii) the optimal policies for power and exponential utility almost have the same trend in the value of the parameter b, σ_r, v, T.

In future research on the asset and liability management problem, it would be interesting to extend our model to the following two aspects. On the one hand, we can consider the ALM problem with stochastic interest rate dynamics in the mean-variance framework and aim to obtain the closed-form solution to the optimal policy and efficient frontier. On the other hand, we can also consider the ALM problem with other stochastic dynamics in the HARA framework and expect to achieve the explicit expression of the optimal policy, for example, Heston's stochastic volatility and CEV model and so on. In those situations, it may be difficult to guess the form of the value function, and it needs us to explore new methodology.

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