Stability and Performance Analysis of Fractional Order Control Systems

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Abstract: Fractional order system model represents the plant more adequately than integer order model. Fractional order controller is naturally the suitable choice for these fractional order models as well as it is widely used for integer order model also.The significance of fractional order control is that it is a generalization of classical control theory. Most of the works in fractional order control systems are in theoretical nature and controller design and implementation in practice is very small. In this paper, stability and performance analysis of fractional order control systems are briefly explained. To show the effectiveness of article, paper demonstrates illustrative design examples. The major purpose of this paper is to draw attention to the non-conventional way of system analysis and its control.

Key–Words: Fractional order systems, fractional order $PI^{\lambda}D^{\mu}$ controller, PID Controller, stability, performance, modelling & simulation.

1 Introduction

Many research and studies on real systems in the field of system identification during last few decades have revealed the systems inherent fractional order dynamic behavior. Hence, using the notion of fractional-order, it may be a step closer to the real world life as the real processes are generally or most likely fractional. This is the reason that the real dynamic systems can be better represented by the non-integer dynamic models. However, for many of them, the fractionality may be very small. A typical example of a fractional order system is the diffusion of heat into a semi-infinite solid, where the heat flow $q(t)$ in nature is equal to the semi-derivative of the temperature $T(t)$ [1]

$$\frac{d^{0.5}T(t)}{dt^{0.5}} = q(t) \qquad (1)$$

Clearly, using an integer order ordinary differential equation (ODE), description for the above system may differ significantly to the actual situation.

The significance of fractional order control system is that it is a generalization of classical control theory which could lead to more adequate modelling and more robust control performance. Despite of this fact, the integer-order controls are still more welcome due to absence of accurate solution methods for fractional order differential equations (FODEs). But recently, many progresses in the analysis of dynamic system modelled by FODEs have been made and approximation of fractional derivatives and integrals can be used in the wide area of fractional order control systems. It is also observed that PID controllers which have been modified using the notion of fractional order integrator and differentiator applied to the integer order or fractional order plant enhance the system control performance. However, their results did not extend to the more general fractional order PID controller for any benchmark system but in this work, we also consider a simple fractional order PID controller design for DC motor to solve the above problems. The main objective of this paper is to investigate the stability and performance of the fractional order control system by illustrating some design examples.

Reference [2, 4, 6, 7] gives the idea of simple tuning formulas for the design of PID controllers. Some MATLAB function files are used in this paper to simulate the fraction order dynamic system using reference [3]. The rest of the article is organized as follows: In section 2, we present a brief introduction of fractional order system and fractional calculus. Section 3 presents the stability analysis of fraction order systems with two illustrative examples. Section

4 deals with performance analysis of fractional order control system with two design examples. Section 5 concludes this paper by some remarks and conclusion.

2 Fundamentals of Fractional Order System

2.1 A Brief Introduction to Fractional Calculus

Fractional calculus is the generalization of integration and differentiation to fractional order fundamental operator $\alpha D_t^\beta f(t)$, where α and t are the limits and $\beta \in R$ is the order of the operation. The continuous integro-differential operator is defined as [3]

$$\alpha D_t^\beta f(t) = \begin{cases} \frac{d^\beta}{dt^\beta} & : \beta > 0, \\ 1 & : \beta = 0, \\ \int_\alpha^t (d\tau)^{-\beta} & : \beta < 0. \end{cases} \quad (2)$$

There are several definitions of fractional integration and differentiation. The most often used are the Grunwald-Letnikov (GL) definition and the Reimann Liouville definition (RL). For a wide class of functions, the two definitions-GL and RL are equivalent. The GL definition [3] is given as:

Definition 1

$$\alpha D_t^\beta f(t) = \lim_{h\to 0} h^{-\beta} \sum_{j=0}^{[\frac{t-\alpha}{h}]} (-1)^j \binom{\beta}{j} f(t-jh), \quad (3)$$

where [.] means the integer part. The RL definition [3] is given as:

Definition 2

$$\alpha D_t^\beta f(t) = \frac{1}{\Gamma(n-\beta)} \frac{d^n}{dt^n} \int_\alpha^t \frac{f(\tau)}{(t-\tau)^{\beta-n+1}} d\tau, \quad (4)$$

for $(n-1 < \beta < n)$ and where $\Gamma(.)$ is the Gamma function.
For many engineering applications, the Laplace transform are often used. The Laplace transform of the GL and RL fractional differointegral, under zero initial conditions for order β is given by

$$\mathcal{L}\left[\alpha D_t^{\pm\beta} f(t); s\right] = s^{\pm\beta} F(s) \quad (5)$$

2.2 Fractional Order System

The fractional-order system is the direct extension of classical integer-order systems. It is obtained from the fractional-order differential equations. A typical n-term linear fractional order differential equation (FODE) in time domain is given by

$$\alpha_n D_t^{\beta_n} y(t) + \cdots + \alpha_1 D_t^{\beta_1} y(t) + \alpha_0 D_t^{\beta_0} y(t) = 0 \quad (6)$$

Consider the control function which acts on the FODE system (6) as follows:

$$\alpha_n D_t^{\beta_n} y(t) + .. + \alpha_1 D_t^{\beta_1} y(t) + \alpha_0 D_t^{\beta_0} y(t) = u(t) \quad (7)$$

On taking Laplace transform of equation (7), we get

$$\alpha_n s^{\beta_n} Y(s) + ..\alpha_1 s^{\beta_1} Y(s) + \alpha_0 s^{\beta_0} Y(s) = U(s) \quad (8)$$

From equation (8), we can obtain a fractional order transfer function as

$$G(s) = \frac{Y(s)}{U(s)} = \frac{1}{\alpha_0 s^{\beta_0} + \alpha_1 s^{\beta_1} + .. + \alpha_n s^{\beta_n}} \quad (9)$$

In general, the fractional-order transfer function (FOTF) of a single variable dynamic system can be defined as

$$G(s) = \frac{b_0 s^{\gamma_0} + b_1 s^{\gamma_1} + \cdots + b_m s^{\gamma_m}}{a_0 s^{\beta_0} + a_1 s^{\beta_1} + \cdots + a_n s^{\beta_n}} \quad (10)$$

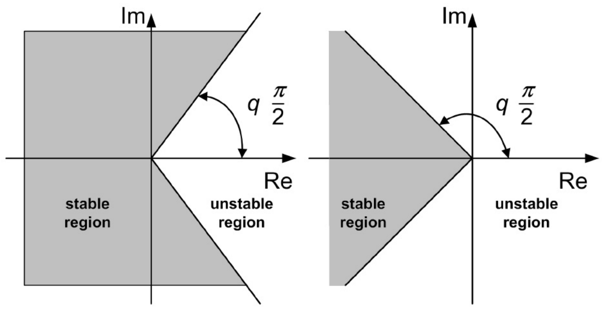
where $b_i (i = 0, 1 \cdots m), a_i (i = 0, 1 \cdots n)$ are constants and $\gamma_i (i = 0, 1 \cdots m), \beta_i (i = 0, 1 \cdots n)$ are arbitrary real or rational numbers and without loss of generality they can be arranged as $\gamma_m > \gamma_{m-1} > \cdots > \gamma_0$ and $\beta_m > \beta_{m-1} > \cdots > \beta_0$.
The incommensurate fractional order system (10) can also be expressed in commensurate form by the multivalued transfer function

$$H(s) = \frac{b_0 + b_1 s^{\frac{1}{\nu}} + \cdots + b_m s^{\frac{m}{\nu}}}{a_0 + a_1 s^{\frac{1}{\nu}} + \cdots + a_n s^{\frac{n}{\nu}}}, (\nu > 1). \quad (11)$$

Note that every fractional order system can be expressed in the form (11) and domain of the $H(s)$ definition is a Riemann surface with ν Riemann sheets.

3 Stability of Fractional Order System

A linear time-invariant system is stable if the roots of the characteristic polynomial are negative or have negative real parts if they are complex conjugate. It means that they are located on the left half of the complex plane. In the fractional-order LTI case, the stability is different from the integer one. Interesting point is that a stable fractional system may have roots in the right half of complex plane (see Fig.1).



(a) $0 < q < 1$ (b) $1 < q < 2$

Figure 1: Stability region of LTI fractional order systems

Theorem 1 *(Matignon's stability theorem) [3]:The fractional transfer function* $G(s) = \frac{N(s)}{D(s)}$ *is stable if and only if* $|arg(\sigma_i)| = q\frac{\pi}{2}$, *where* $\sigma = s^q, (0 < q < 2)$ *with* $\forall \sigma_i \in C, i^{th}$ *root of* $D(\sigma) = 0$.

Remark 1 *When s =0 is a single root of D(s), the system cannot be stable.*

For theorem 1, the stability region suggested by Fig.(1) tends to the whole s-plane when $q = 0$, corresponds to the Routh-Hurwitz stability when $q = 1$ and tends to the negative real axis when $q = 2$.

It should be noted that, only the denominator is meaningful in stability assessment and the numerator does not affect the stability of a FOTF. The stability of fractional order system can be analyzed in other way also. Consider the characteristic equation of a general fractional order system in the form as:

$$\alpha_0 s^{\beta_0} + \alpha_1 s^{\beta_1} + \cdots + \alpha_n s^{\beta_n} = \sum_{i=0}^{n} \alpha_i s^{\beta_i} = 0. \quad (12)$$

For $\beta_i = \frac{\nu_i}{\nu}$, we transform the equation (12) into σ-plane:

$$\sum_{i=0}^{n} \alpha_i s^{\frac{\nu_i}{\nu}} = \sum_{i=0}^{n} \alpha_i \sigma^{\nu_i} = 0 \quad (13)$$

where $\sigma = s^{\frac{k}{m}}$ and m is the least common multiple of ν.

For given α_i, if the absolute phase of all roots of transform equation (13) is $|\phi_\sigma| = |arg(\sigma)|$, we can summarize the following facts of stability for fractional order systems :

1. The condition for stability is $\frac{\pi}{2m} < |arg(\sigma)| < \frac{\pi}{m}$.
2. The condition for oscillation is $|arg(\sigma)| = \frac{\pi}{2m}$.

Otherwise the system is unstable.

3.1 Illustrative Examples

To show the effectiveness of stability concept for fractional order system, we have demonstrated two illustrative examples.

Example 1

Consider the fractional order heater system [4] given by

$$G(s) = \frac{1}{0.8s^{2.2} + 0.5s^{0.9} + 1} \quad (14)$$

The characteristic equation of the system (14) can be written as:

$$D(s) = 0.8s^{\frac{22}{10}} + 0.5s^{\frac{9}{10}} + 1 = 0 \quad (15)$$

Now we write the transformed equation for equation (15) in σ-plane as

$$D(\sigma) = 0.8\sigma^{22} + 0.5\sigma^{9} + 1 = 0 \quad (16)$$

The equation (16) is solved using MATLAB function `solve( )` and we obtained the following roots:

$$\begin{cases} \sigma_{1,2} = 0.3080 \pm 0.9772i; |arg(\sigma_{1,2})| = 3.023 \\ \sigma_{3,4} = 0.5243 \pm 0.8359i; |arg(\sigma_{3,4})| = 1.010 \\ \sigma_{5,6} = -0.9297 \pm 0.4414i; |arg(\sigma_{5,6})| = 2.698 \\ \sigma_{7,8} = -0.0254 \pm 1.0111i; |arg(\sigma_{7,8})| = 1.595 \\ \sigma_{9,10} = -0.2596 \pm 0.9625i; |arg(\sigma_{9,10})| = 1.834 \\ \sigma_{11,12} = -0.9970 \pm 0.1182i; |arg(\sigma_{11,12})| = 3.023 \\ \sigma_{13,14} = 0.7793 \pm 0.6795i; |arg(\sigma_{13,14})| = 0.717 \\ \sigma_{15,16} = -0.5661 \pm 0.8633i; |arg(\sigma_{15,16})| = 2.151 \\ \sigma_{17,18} = -0.7465 \pm 0.6420i; |arg(\sigma_{17,18})| = 2.431 \\ \sigma_{19,20} = 1.0045 \pm 0.1684i; |arg(\sigma_{19,20})| = 0.1661 \\ \sigma_{21,22} = 0.9084 \pm 0.3960i; |arg(\sigma_{21,22})| = 0.411 \end{cases} \quad (17)$$

Considering all roots listed in (17) of characteristic equation (16), we find that complex conjugate roots $\sigma_{19,20} = 1.0045 \pm 0.1684i; |arg(\sigma_{19,20})| = 0.1661$, satisfy the stability condition $\frac{-\pi}{m} < arg(\sigma) < \frac{\pi}{m} \Rightarrow -0.3 < 0.1661 < 0.3$ and $|arg(\sigma)| > \frac{\pi}{2m} \Rightarrow 0.1661 > 0.157$. Hence it shows that fractional order system $G(s)$ is stable.

MATLAB function `isstable` defined under `fotf class` [3] can also be used to test approximately the stability of a given fractional order transfer function model (14). The returned argument say K is the stability of the system, with 1 for stable and 0 for unstable. Using this `isstable` function, the denominator of $G(s)$, $0.8s^{2.2} + 0.5s^{0.9} + 1$ is checked and

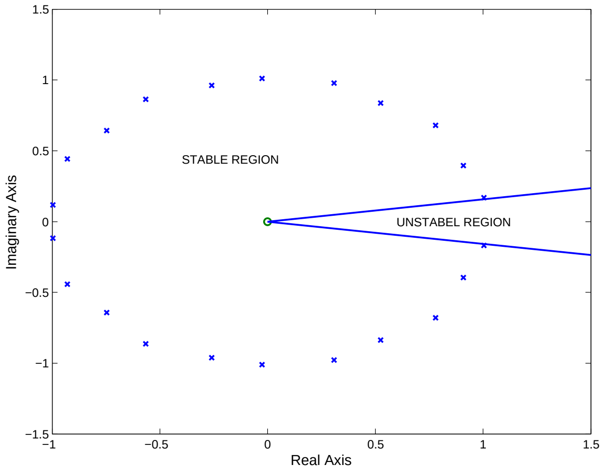


Figure 2: Poles position in complex σ-plane

it is found that $K = 1$, indicating the system is stable, with $q = 0.1$. Stability region for given fractional order system (14) is shown in Fig.(2). The region of stability depends on the value of q. Since q is 0.1, the angle is around 9°. Here we can see that $G(s)$ has more stability region. Now it can also be concluded that system is stable even if it's poles lie on the right side of the complex plane.

Example 2

Consider fraction order system $P(s) = \frac{1}{-1.5s^{0.5}+1}$ The characteristic equation of $P(s)$ can be written as

$$D(s) = -1.5s^{\frac{5}{10}} + 1 = 0 \tag{18}$$

The transformed equation of (18) in σ-plane for $\sigma = s^{\frac{1}{10}}$ and $m = 10$ is given by

$$D(\sigma) = -1.5\sigma^5 + 1 = 0 \tag{19}$$

The roots are: $\sigma_1 = 0.9221, \sigma_{2,3} = -0.7460 \pm 0.5420i$ and $\sigma_{4,5} = 0.2849 \pm 0.8769i$.

Here $|arg(\sigma_{2,3})| = 0.628, |arg(\sigma_{4,5})| = 1.25$.

Thus no roots of equation (19) satisfy the stability condition $\frac{-\pi}{m} < arg(\sigma) < \frac{\pi}{m}$ and $|arg(\sigma)| > \frac{\pi}{2m}$. It indicates that the given system $P(s)$ is unstable.

4 Fractional Order Control

Fraction order control is the non-conventional way of robust control based on fractional order derivative. Most of the works in fractional order control systems are in theoretical nature and controller design and implementation in practice is very small. In this article, main objective is to apply the fractional order control (FOC) to analyze the system control performance. In theory, the control system can include both the fractional order plant to be controlled and fractional order controller. However, in control practice, more common is to consider the fractional order controller only. Here we demonstrate two scenarios - (a) Integer order plant is being controlled by fractional order controller and (b) fractional plant is being controlled by fractional order controller.

The fractional-order $PI^{\lambda}D^{\mu}$ controller is a generalization of PID controller with integrator of real order λ and differentiator of real order μ. The transfer function of fractional order PID controller is given by

$$C(s) = K_P + \frac{K_I}{s^{\lambda}} + K_D s^{\mu}, (\lambda, \mu > 0) \tag{20}$$

Taking $\lambda = 1$ and $\mu = 1$, we obtain a classical PID controller. If $\lambda = 0$ and $\mu = 0$, we can obtain a PD^{μ} and PI^{λ} controller respectively. All these type of controllers are particular case of the $PI^{\lambda}D^{\mu}$ controller, which is more flexible and gives an opportunity to better adjust the dynamical properties of the fractional-order control system

4.1 Fractional-Order Controller Design for Integer Order Plant

Consider the transfer function model of DC motor [5], given by

$$G_{DCM}(s) = \frac{0.08}{s(0.05s + 1)} \tag{21}$$

The feedback control loop is depicted in Fig.3, where $C(s)$ is the transfer function of controller and $G_{DCM}(s)$ is transfer function of motor. The applied voltage V_a control the angular velocity ω of the system.

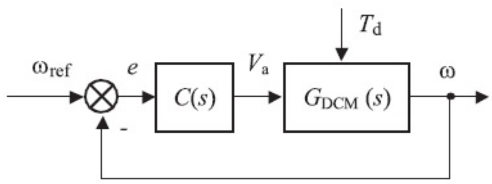


Figure 3: Feedback control loop

The several methods and tuning technique for fractional order PID parameters were developed during last decade [2, 4, 6, 7]. They are based on various approaches. In this work, we select K_P, K_I and K_D values for fractional order PID controller of DC motor

only in two steps:
Step-01:The design of K_P
Proportional gain K_P is related to the static error E_t [%], settling time T_r [sec.] and overshoot P_r [%]. In general, K_P can be set through

$$K_P \geq \left(\frac{100}{E_t}\right) \tag{22}$$

To find minimum static error, we select proportional gain $K_P = 10$.
Step-02:The design of K_D, μ, K_I and λ
To select these values for FOC design, we use following controller synthesis scheme:
First we define the phase margin of the controlled system as

$$\phi_m = \angle[C(s)G_{DCM}(s)] + \pi \tag{23}$$

The equation (23) can be accomplished by controller $C(s) = K(s)$ of the form

$$\begin{cases} K(s) = k_1 \frac{k_2 s+1}{s^\nu} \\ k_1 = \frac{1}{K_{DCM}}, k_2 = \tau \end{cases} \tag{24}$$

where $\tau = 0.05$ is the time constant and $K_{DCM} = 0.08$ is constant gain of the DC motor respectively. Such controller (24) gives a constant phase margin. Now using equation (23) and (24), we have

$$\begin{cases} \phi_m = arg\,[C(j\omega)G_{DCM}(j\omega)] + \pi \\ = arg\left[\frac{k_1 K_{DCM}}{(j\omega)^{(1+\nu)}}\right] + \pi \\ = arg\left[(j\omega)^{-(1+\nu)}\right] + \pi \\ = \pi - (1+\nu)\frac{\pi}{2} \end{cases} \tag{25}$$

If we fix phase margin $\phi_m \geq 60°$ for the controlled object DC motor, we find the constants $\nu = 0.3$ from equation (25). The other desired constant values $k_1 = 12.5, k_2 = 0.05$ can be obtained from equation (24).

Now using these constants in equation (24), we can obtain a fractional $I^\lambda D^\mu$ controller, which is a particular case of $PI^\lambda D^\mu$ controller and has the form

$$K(s) = 0.625 s^{0.7} + \frac{12.5}{s^{0.3}} \tag{26}$$

where $K_D = 0.625, K_I = 12.5, \mu = 0.7, \lambda = 0.3$
Finally,on substituting the values of $K_P = 10$ from *step*-01 and $K_D = 0.625, K_I = 12.5, \mu = 0.7, \lambda = 0.3$ from *step*-02 in equation (20), we obtain the following fractional order PID controller transfer function:

$$C(s) = 10 + \frac{12.5}{s^{0.3}} + 0.625 s^{0.7} \tag{27}$$

Now the transfer function of the feedback control loop with the fractional order PID controller (27) is obtained as

$$\begin{cases} G_{cl} = \frac{G_o(s)}{1+G_o(s)} = \frac{G_{DCM}(s)C(s)}{1+G_{DCM}(s)C(s)} \\ = \frac{0.05s+0.8s^{0.3}+1}{0.05s^{2.3}+s^{1.3}+0.05s+0.8s^{0.3}+1} \end{cases} \tag{28}$$

where $G_o(s)$ is the transfer function of open control loop with $G_o(s) = \frac{0.05s+0.8s^{0.3}+1}{0.05s^{2.3}+s^{1.3}}$.

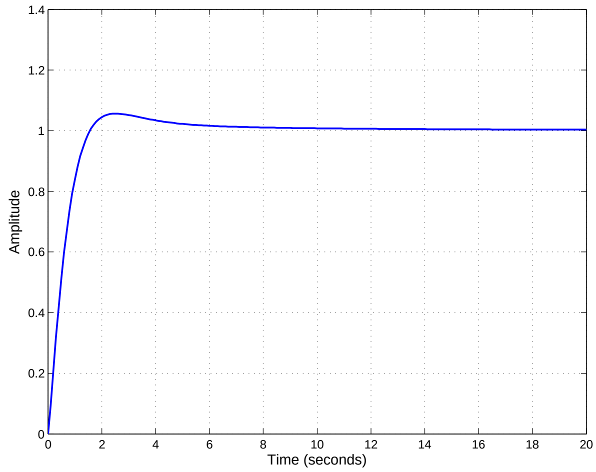


Figure 4: Step response of DC motor with fractional order PID controller

The fractional order DC motor feedback system (28) is simulated in MATLAB environment under `fotf class` using reference [3]. The step response for fraction order feedback system G_{cl} is obtained and it is shown in Fig.(4). The design exhibits a very negligible overshoot and effectively achieves its steady state within 5 second only.The bode diagram of the controlled model is also presented in Fig.(5). It can be seen that phase margin $\phi_m \approx 74°(> 60°)$ and gain margin is infinity, which satisfy our desired specifications.

4.2 Fractional-Order Controller Design for Fractional Order Plant

We consider the following fractional order plant model given in [4]:

$$G_F(s) = \frac{1}{0.8s^{2.2} + 0.5s^{0.9} + 1} \tag{29}$$

Using least-squares method, the following approximated integer order model corresponding to (29) was obtained in [4]:

$$G_I(s) = \frac{1}{0.714s^2 + 0.2313s + 1} \tag{30}$$

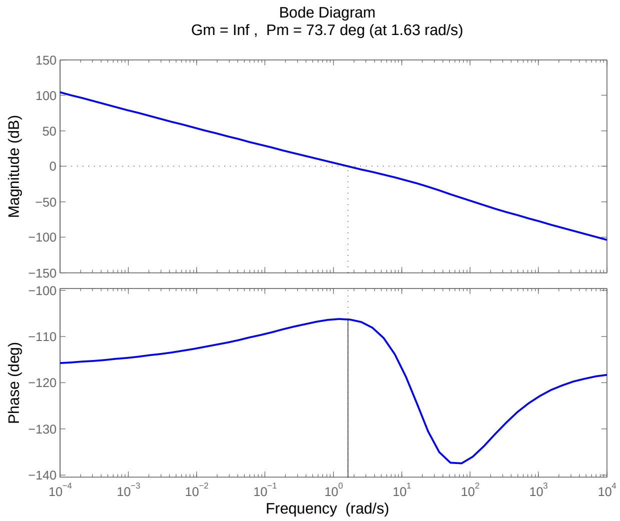


Figure 5: Bode diagram of DC motor with fractional order PID controller

The comparison of unit step response of the systems described by (29) and (30) are shown in Fig.(6).

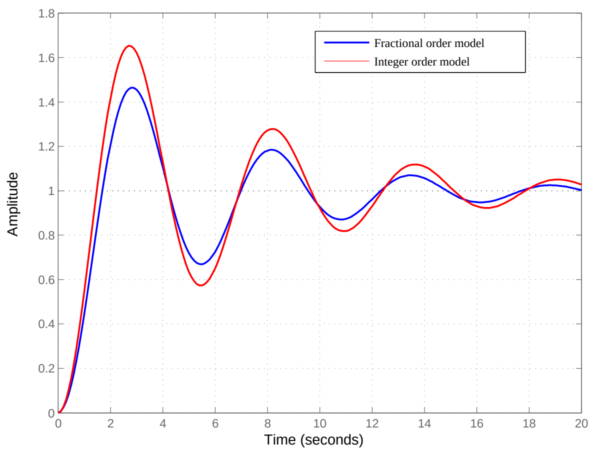


Figure 6: Comparison of unity step response of the integer order model and the fractional order model

The integer order PD controller and the fractional order PD^μ were designed in [4]. The integer order PD is given by

$$G_c(s) = 20.5 + 2.7343s \tag{31}$$

while the fractional order PD^μ is characterized by the fractional order transfer function [4]

$$G_c = 20.5 + 3.7343s^{1.15} \tag{32}$$

In Fig.(7), comparison of unit step response of the closed loop fractional order system controlled by fractional order PD^μ controller and integer order PD controller is presented.

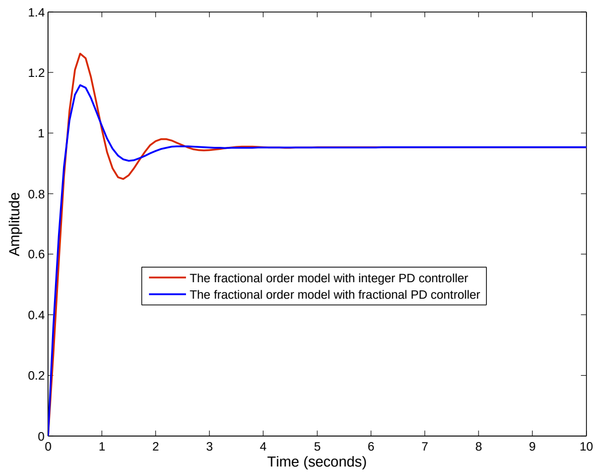


Figure 7: Comparison of unity step response of the closed loop fractional order system with integer order PD controller and with fractional order PD^μ controller

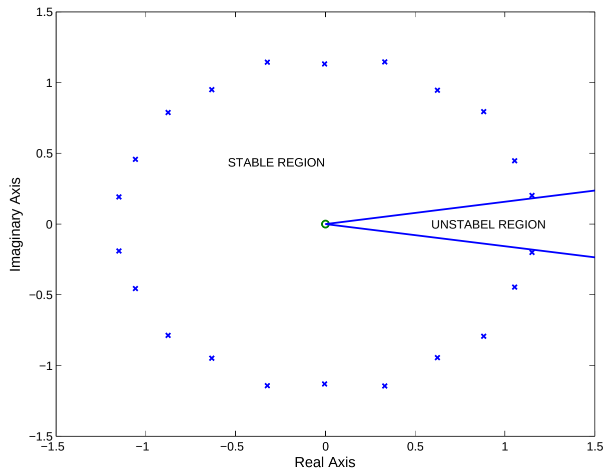


Figure 8: Stability region of fractional order system controlled by integer order PD controller

Stability analysis is also investigated for the designed fractional order control system. The simulation results for the stability of fractional order system controlled by integer order controller and fractional order controller are depicted in Fig.(8) and Fig.(9) respectively.

The conclusion is that the use of the fractional order controller leads to an improvement of performance of fractional order system (see Fig.(7)) and increases the stability region of the system (see Fig.(8) and Fig.(9)).

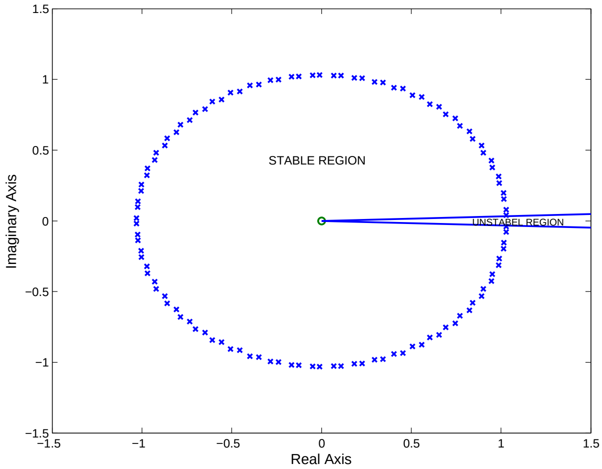


Figure 9: Stability region of fractional order system controlled by fractional order PD^{μ} controller

5 Conclusion

In this paper, stability and performance analysis of fractional order control systems is investigated. The basic ideas and technical formulations for the analysis of fractional order control systems are presented. Some illustrative design examples with simulation results have been demonstrated. The major purpose of this paper is to draw attention to the non-conventional way of system analysis and its control. We believe that fractional order control can benefit control engineering practitioners in number of ways.

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References:

[1] I. Podlubny, L. Dorcak, and J. Misanek, Application of fractional-order derivatives to calculation of heat load intensity change in blast furnace walls, *Transactions of Tech. Univ. of Kosice*, vol. 5, no. 5, 1995, pp.137-144.

[2] Monje C. A.,Vinagre B. M., Feliu V., Chen Y.Q., Tuning and Auto-Tuning of Fractional Order Controllers for Industry Application, *Control Engineering Practice*, 2008, pp.798-812.

[3] Y. Q. Chen, Ivo Petras and Dingyu Xue , Fractional Order Control: A Tutorial, *American Control conference*, June 10-12, 2009

[4] I. Podlubny, Fractional-order systems and fractional-order controllers, *The Academy of Sciences Institute of Experimental Physics*, Kosice, Slovak Republic, 1994.

[5] Dorf R.C., Bishop R.H., Modern Control System, *Addison-Wesley*, New York, 1990

[6] I. Podlubny, Fractional-order systems and $PI^{\lambda}D^{\mu}$-controllers, *IEEE Trans. Automatic Control*, vol. 44,1999, pp. 208-214.

[7] Petras I., The fractional-Order Controllers: Methods for their Synthesis and Application, *Journal of Electrical Engineering*, No. 9-10, 1999, pp.284-288.