An existence result for non-convex optimal control problems

CLARA CARLOTA
Cima-ue
rua Romão Ramalho 59, P-7000-671 Évora
PORTUGAL
ccarlota@uevora.pt

SÍLVIA CHÁ
Cima-ue
rua Romão Ramalho 59, P-7000-671 Évora
PORTUGAL
silviaaccha@gmail.com

Abstract: We present new existence results of solutions to autonomous Lagrange optimal control problems, without assuming superlinear growth, nor convexity, on the lagrangian.

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1 Introduction

Optimal control theory deals with the problem of how to control, in the best possible way, the state of a system that changes along time with the aim of reaching a given target state, while respecting given dynamics and given constraints. More precisely, our research deals with optimal control problems in which "best" means that one wishes to minimize an integral; while the functions in competition are subject to given pointwise constraints in their states. For details on optimal control theory we refer e.g. the books [7], [8], [11], [10], [13], [17], [20], [35], [36], [37], [38], and references therein. For recent real-life applications of optimal control theory to solve diverse problems of several areas as stochastic optimization, economics, engineering, physics, medicine, biology, ..., see e.g. [32], [33], [12], [22], [1], [16], [3], [21], [18], [34], [19], [23], [29], and references therein.

In this paper we are concerned with existence of solutions to the autonomous Lagrange optimal control problem $(\mathcal{OCP})$, i.e. the problem of minimizing the cost (or objective) functional

$$J(x,u) := \int_a^b f_0(x(t), u(t))\, dt\,, \tag{1}$$

over all pairs $(x(\cdot), u(\cdot))$ whose trajectories

$$x(\cdot) \in W^{1,1}([a,b], \mathbb{R}^n) \tag{2}$$

satisfy the state constraint

$$x(t) \in \Omega \subset \mathbb{R}^n \ \ \forall t \in [a,b]\,, \tag{3}$$

reach the terminal state

$$x(b) = B \tag{4}$$

and obey the dynamics given by the nonlinear, control-affine, ordinary differential equation

$$\begin{aligned} x'(t) &= f(x(t), u(t)) \\ &:= A_0(x(t)) + B_0(x(t))\, u(t) \\ &\quad\ \text{for a.e. } t \in [a,b] \\ x(a) &= A\,, \end{aligned} \tag{5}$$

where the controls

$$u : [a,b] \to \mathbb{R}^m \tag{6}$$

are measurable functions satisfying the state-dependent constraint :

$$u(t) \in U(x(t)) \ \text{ a.e. on } \ [a,b]\,. \tag{7}$$

Here U is a multifunction defined in Ω with values $U(s)$ in the class $2^{\mathbb{R}^m} \setminus \emptyset$ of all nonempty subsets of $\mathbb{R}^m$.

Denote by $\mathcal{Y}_{A,B}^{n,m}$ the class of admissible pairs $(x(\cdot), u(\cdot))$ for $(\mathcal{OCP})$, i.e.

$$\mathcal{Y}_{A,B}^{n,m} := \{(x(\cdot), u(\cdot)) \text{ satisfying (2) to (7)}\}\,.$$

We aim at find out whether there exist, among these admissible pairs, some which are optimal, in the sense of minimizing the integral (1); and we will call any such optimal pair a *true solution* to the $(\mathcal{OCP})$.

Assuming $\mathcal{Y}_{A,B}^{n,m} \neq \emptyset$ and Ω closed, together with continuity of $f_0(\cdot,\cdot)$ and $f(\cdot,\cdot)$ on $M := \{(s,u) : s \in \Omega,\ u \in U(s)\}$, then the simplest case — in which the classical Tonelli's Direct Method can be applied (see e.g. [7]) to prove existence of true solutions — is the so-called convex-coercive case, in which the following hypotheses are satisfied :

$|f(s,u)| \to \infty$ uniformly on Ω as $|u| \to \infty$, $f_0(s,\cdot)$ is convex $\forall\, s \in \Omega$ and coercive:

$$f_0(s,u) \geq \theta(|f(s,u)|) \ \ \forall\, (s,u) \in M,$$

where $\theta : [0,\infty) \to \mathbb{R}$ is bounded from below with $\theta(r)/r \to \infty$ as $r \to \infty$.

Our main purpose is to prove new existence results without imposing any kind of convexity nor coercivity — in which case, as is well known, an optimal pair $(x(\cdot), u(\cdot))$ may fail to exist.

Concerning previous theoretical results of existence for the above non-convex autonomous Lagrange $(\mathcal{OCP})$, with general $f_0(s,u)$ we are aware only of [28] (for one-dimensional space problems, i.e. $n = 1$) and [24] (with $f_0(\cdot,\cdot)$ continuous and $f_0(\cdot,u)$ of class C^1). To the best of our knowledge, the other existence results achieved until now are for the sum case $f_0(s,u) = g(s) + h(u)$ with $g(\cdot)$ concave (see e.g. [7], [30], [14], [15] and references therein) or the case $f_0(s,u) = \sum_{i=1}^{\nu} c_i(s)\,\phi_i(u)$, with $c_i(\cdot)$ continuous and $\phi_i(\cdot)$ of class C^1, proved in [27].

Compared with these previous results, our improvements mainly concern our much weak assumptions on the lagrangian $f_0(\cdot,\cdot)$. Their applicability can be readily checked: see the last section of this paper for concrete numerical examples where, as far as we know, none of the previous available results can be applied.

The above optimal control problem can be reformulated as a calculus of variations problem as follows (see e.g. [7], [10], [26]). Define

$$\widetilde{Q}(s) := \begin{cases} \left\{ \begin{array}{l} (v, v_0) \in \mathbb{R}^n \times \mathbb{R} : \\ v_0 \geq f_0(s,u), \\ v = f(s,u), \\ u \in U(s) \end{array} \right\} \\ \text{if } s \in \Omega \\ \\ \emptyset \ \text{otherwise} \end{cases} \tag{8}$$

and the associated lagrangian

$$\begin{aligned} &L : \mathbb{R}^n \times \mathbb{R}^n \to [0,\infty] \\ &L(s,v) := \inf\left\{ v_0 : \ (v, v_0) \in \widetilde{Q}(s) \right\}. \end{aligned} \tag{9}$$

Then, under adequate hypotheses, the study of the optimal control problem is equivalent to the study of the calculus of variations problem which consists in minimizing the integral

$$I(x) := \int_a^b L\left(x(t), x'(t)\right)\, dt$$

on

$$\mathcal{X}^n_{A,B} := \left\{ \begin{array}{l} x(\cdot) \in W^{1,1}([a,b], \mathbb{R}^n) : \\ x(a) = A, \ \ x(b) = B \end{array} \right\},$$

in the sense that there exists a solution to $(\mathcal{OCP})$ if and only if there exists a minimizer for $I(\cdot)$. Notice that the constraints (5) and (7) above are implicitly included in the calculus of variations problem since, by setting, for $s \in \Omega$,

$$Q(s) := \{ v \in \mathbb{R}^n : \ \ v = f(s,u), \ u \in U(s) \},$$

we have

$$L(s,v) = \infty \ \text{ whenever } \ v \notin Q(s).$$

Therefore, our strategy to prove existence of solutions to $(\mathcal{OCP})$, under our more general hypotheses than has been previously considered, will consist in the following steps:

Step 1. Since we are not assuming any kind of convexity on $f_0(\cdot,\cdot)$, we start by considering the associated convexified optimal control problem $(\mathcal{OCP}_c)$:

minimize

$$J_c(x,u) := \textstyle\int_a^b f_0^{**}(x(t), u(t))\, dt,$$

where $f_0^{**}(s,\cdot)$ is the bipolar of $f_0(s,\cdot)$ (i.e. — see e.g. [31] — the pointwise supremum of all affine minorants of $f_0(s,\cdot)$), over all pairs $(x(\cdot), u(\cdot))$, with $x(\cdot) \in \mathcal{X}^n_{A,B}$ and $u : [a,b] \to \mathbb{R}^m$ measurable, satisfying (3), (5) and

$$u(t) \in co\,U(x(t)) \ \text{ for a.e. } \ t \in [a,b],$$

where $co\,U(s)$ stands for the convex hull of the set $U(s)$. Denote this class, of admissible pairs $(x(\cdot), u(\cdot))$ for $(\mathcal{OCP}_c)$, by $\mathcal{Z}^{n,m}_{A,B}$ and assume, or prove, existence of a *relaxed solution* $(\overline{x}_c(\cdot), \overline{u}_c(\cdot))$, i.e. a solution to this problem.

Step 2. Reformulate $(\mathcal{OCP}_c)$ as a (convex) Lagrange problem of the calculus of variations $(\mathcal{CVP}_c)$:

minimize

$$I_c(x) := \textstyle\int_a^b L^{**}(x(t), x'(t))\, dt$$

on $\mathcal{X}^n_{A,B}$

in such a way that $\overline{x}_c(\cdot)$ is itself a minimizer to this integral. (Here $L^{**}(s,\cdot)$ will be the bipolar of the function $L(s,\cdot)$ defined in (9).)

Step 3. Prove existence of a solution $\overline{x}(\cdot)$ to the non-convex problem $(\mathcal{CVP})$:

$$\begin{aligned}&\text{minimize}\\ &I(x) := \int_a^b L(x(t), x'(t))\, dt\\ &\text{on } \mathcal{X}^n_{A,B}.\end{aligned}$$

Step 4. Use such $\overline{x}(\cdot)$ to obtain a solution $(\overline{x}(\cdot), \overline{u}(\cdot))$ to the non-convex $(\mathcal{OCP})$.

This paper is organized as follows. In section 2 we describe our results on existence of minimizers for the non-convex problem of the calculus of variations $(\mathcal{CVP})$ which we will use in the intermediated Step 3; while section 3 is devoted to our Basic Hypotheses and to the statement, and proof, of auxiliary results. The main results of this paper, which guarantee existence of solution to the $(\mathcal{OCP})$, appear in section 4. Finally, in section 5 we present numerical examples of application.

2 The Lagrange problem of the calculus of variations

We begin this section with the definition of almost convex function.

Definition 1 *Given a function* $L : \mathbb{R}^n \times \mathbb{R}^n \to (-\infty, \infty]$, *we say that* $L(s, \cdot)$ *is almost convex provided*

at each v *where* $L^{**}(s,v) < L(s,v)$,

$$\begin{aligned}&\exists\, \lambda = \lambda(s,v) \in [0,1] \;\; \exists\, \Lambda = \Lambda(s,v) \in [1,\infty)\\ &\exists\, \alpha = \alpha(s,v) \in [0,1]\ :\end{aligned}$$

$$\begin{aligned}&L^{**}(s,v) = (1-\alpha)\, L(s, \lambda v) + \alpha\, L(s, \Lambda v)\\ &v = (1-\alpha)\,(\lambda v) + \alpha\,(\Lambda v)\end{aligned}$$

(for completeness, we set $\lambda = 1 = \Lambda = \alpha$ *where* $L^{**}(s,v) = L(s,v)$ *— in particular at* $v = 0$ *— and use the convention* $0 \cdot \infty = 0$*).*

Notice that if $L(s, \cdot)$ is convex and lower semicontinuous (lsc) then $L(s, \cdot)$ is almost convex, since, in that case, $L^{**}(s, \cdot) = L(s, \cdot)$.

Proposition 2 *Let* $L : \mathbb{R}^n \times \mathbb{R}^n \to [0,\infty]$ *be a Borel function having* $L^{**}(\cdot,\cdot)$ *Borel and* $L(s,\cdot)$ *almost convex lsc* $\forall\, s$. *Then for any* $\overline{x}_c(\cdot) \in W^{1,1}([a,b], \mathbb{R}^n)$ *there exist measurable functions*

$$\begin{aligned}\lambda_1 &= \lambda_1(\overline{x}_c, \overline{x}_c') : [a,b] \to [0,1],\\ \lambda_2 &= \lambda_2(\overline{x}_c, \overline{x}_c') : [a,b] \to [1,\infty)\end{aligned} \tag{10}$$

such that, at a.e. $t \in [a,b]$,

$$\begin{aligned}&L^{**}(\overline{x}_c(t), \overline{x}_c'(t)) =\\ &(1-p(t))\, L(\overline{x}_c(t), \lambda_1(t)\, \overline{x}_c'(t)) +\\ &+ p(t)\, L(\overline{x}_c(t), \lambda_2(t)\, \overline{x}_c'(t))\end{aligned}$$

with $p : [a,b] \to [0,1]$,

$$p(t) := \begin{cases} 1 & \text{if } \lambda_1(t) = \lambda_2(t) = 1\\ \frac{1-\lambda_1(t)}{\lambda_2(t)-\lambda_1(t)} & \text{otherwise}.\end{cases}$$

This concept of almost convexity was introduced, for multifunctions, in the paper by A. Cellina & A. Ornelas [6] to prove, using reparametrizations, existence of solutions to non-convex upper semicontinuous differential inclusions and to time optimal control problems. On the other hand, almost convex lagrangians were first defined in [5] to prove results of existence of solutions to non-convex problems of the calculus of variations, using bimonotone reparametrizations. The concept of bimonotone minimizer was born in the paper by A. Ornelas [25].

Proposition 3 *(See [5, Cor. 9].) Let* $L : \mathbb{R}^n \times \mathbb{R}^n \to [0,\infty]$ *be a Borel function having* $L^{**}(\cdot,\cdot)$ *Borel,* $L(\cdot, 0)$ *lsc and* $L(s,\cdot)$ *almost convex lsc* $\forall\, s$.

If there exists a minimizer $\overline{x}_c(\cdot)$ *to the convexified integral* $I_c(\cdot)$ *then also exists a minimizer* $\overline{x}(\cdot)$ *to the non-convex integral* $I(\cdot)$.

For the superlinear case, we have the

Proposition 4 *(See [5, Th. 5].) Let* $L : \mathbb{R}^n \times \mathbb{R}^n \to [0,\infty]$ *be a lsc function with superlinear growth, i.e.*

$$\begin{aligned}&L(s,v) \geq \theta(|v|) \;\; \forall\, (s,v)\\ &\text{with } \tfrac{\theta(r)}{r} \to \infty \text{ as } r \to \infty,\end{aligned}$$

having $L(s,\cdot)$ *almost convex* $\forall\, s$.

Then the non-convex integral $I(\cdot)$ *has a minimizer* $\overline{x}(\cdot)$.

In our recent paper [4] we have proved that the hypothesis of $L(\cdot, 0)$ to be lsc, in Proposition 3, is not needed if, for instance, the set (see (10))

$$E_0 := \{t \in [a,b] : \ \lambda_1(t) = 0\}$$

has zero measure. Indeed, much more generally,

Proposition 5 *(See [4, Th. 7].) Assume $L : \mathbb{R}^n \times \mathbb{R}^n \to [0,\infty]$ is a Borel function having $L^{**}(\cdot,\cdot)$ Borel and $L(s,\cdot)$ almost convex lsc for each $s \in \mathbb{R}^n$.*

Admit also the validity of the following Extra Hypothesis (EH):

there exists a minimizer $\overline{x}_c(\cdot)$ to $I_c(\cdot)$ for which

either the set
$E_0 := \{t \in [a,b] : \ \lambda_1(t) = 0\}$
has zero measure

or else:
$\overline{E_0} \setminus E_0$ *is a null set (with* $\overline{E_0}$ *the closure of* E_0*)*
and
$\exists\, t_c \in [a,b]$ *such that*
$L(\overline{x}_c(t_c), 0) \le L(\overline{x}_c(t), 0) \ \ \forall\, t \in [a,b]$.

Then there exists a minimizer $\overline{x}(\cdot)$ to the nonconvex integral $I(\cdot)$.

Remark 6 *In Propositions 3, 4 and 5, we have*

$$\min_{x(\cdot) \in \mathcal{X}^n_{A,B}} I(x(\cdot)) = \min_{x(\cdot) \in \mathcal{X}^n_{A,B}} I_c(x(\cdot)). \qquad (11)$$

Notice that an essential feature of the above results, which allow us to apply them in order to prove existence of solutions to the optimal control problem, is that the lagrangian function $L(\cdot,\cdot)$ may freely assume the ∞ value.

3 Basic Hypotheses and auxiliary results

Denote by $gph(U)$ the graph of U:

$$gph(U) := \left\{ \begin{array}{l} (s,u) \in \mathbb{R}^n \times \mathbb{R}^m : \\ s \in \Omega, \ u \in U(s) \end{array} \right\}$$

and by $co\,U$ the multifunction

$$co\,U : \Omega \to 2^{\mathbb{R}^m} \setminus \emptyset$$
$$(co\,U)(s) := co\,U(s).$$

We will set

$$f_0(s,u) := \infty \ \text{ for } \ (s,u) \notin gph(U)$$

and assume the Basic Hypotheses:

(BH1) $\Omega \subset \mathbb{R}^n$ is closed

(BH2) $U : \Omega \to 2^{\mathbb{R}^m} \setminus \emptyset$ is such that

(BH2.1) $gph(U)$ is closed
(BH2.2) $gph(co\,U)$ is closed

(BH3) $f_0 : gph(U) \to [0,\infty)$ is lsc

(BH4) $f_0^{**} : gph(co\,U) \to [0,\infty)$ is lsc

(BH5) $f : gph(co\,U) \to \mathbb{R}^n$
$f(s,u) = A_0(s) + B_0(s)\,u$,
where, for every $s \in \Omega$,
$A_0(s)$ is a $n \times 1$ matrix
$B_0(s)$ is a $n \times m$ matrix
$A_0(\cdot)_{i1}$, $B_0(\cdot)_{ij} : \Omega \to \mathbb{R}$
are continuous functions
$(i = 1, ..., n;\ j = 1, ..., m)$

(BH6) for every $s \in \Omega$, the sets

$$f(s, co\,U(s))$$

$$\left\{ \begin{array}{l} (f(s,u), f_0(s,u) + \mu) : \\ u \in U(s), \ \ \mu \ge 0 \end{array} \right\}$$

$$\left\{ \begin{array}{l} (f(s,u), f_0^{**}(s,u) + \mu) : \\ u \in co\,U(s), \ \ \mu \ge 0 \end{array} \right\}$$

are closed.

Proposition 7 *If the Basic Hypotheses $(BH2) - (BH6)$ hold true then $L : \mathbb{R}^n \times \mathbb{R}^n \to [0,\infty]$ defined in (9) is a well-defined Borel function having*

$L^{**}(\cdot,\cdot)$ *Borel and* $L(s,\cdot)$ *lsc* $\forall s$. *Moreover,*

$$\begin{aligned} & L(s,v) \\ = \; & \inf\{f_0(s,u): \; u \in H(s,v)\} \\ = \; & \begin{cases} \min\left\{\begin{array}{l} f_0(s,u): \\ u \in H(s,v) \end{array}\right\} \\ \textit{if } s \in \Omega \textit{ and} \\ v \in f(s,U(s)) \\ \\ \infty \; \textit{otherwise} \end{cases} \end{aligned} \tag{12}$$

where

$$H(s,v) := \begin{cases} \left\{\begin{array}{l} u \in U(s): \\ v = f(s,u) \end{array}\right\} \\ \textit{if } s \in \Omega \\ \\ \emptyset \; \textit{otherwise}. \end{cases} \tag{13}$$

Also,

$$\begin{aligned} & L^{**}(s,v) \\ = \; & \inf\{f_0^{**}(s,u): \; u \in H_0(s,v)\} \\ = \; & \begin{cases} \min\left\{\begin{array}{l} f_0^{**}(s,u): \\ u \in H_0(s,v) \end{array}\right\} \\ \textit{if } s \in \Omega \textit{ and} \\ v \in f(s, co\,U(s)) \\ \\ \infty \; \textit{otherwise} \end{cases} \end{aligned} \tag{14}$$

where

$$H_0(s,v) := \begin{cases} \left\{\begin{array}{l} u \in co\,U(s): \\ v = f(s,u) \end{array}\right\} \\ \textit{if } s \in \Omega \\ \\ \emptyset \; \textit{otherwise}. \end{cases} \tag{15}$$

Proof: The equalities in (12) hold true since, by $(BH6)$, for every $s \in \Omega$ the set (see (8))

$$\widetilde{Q}(s) = \left\{\begin{array}{l} (f(s,u), f_0(s,u) + \mu): \\ u \in U(s), \;\; \mu \geq 0 \end{array}\right\}$$

is closed. Moreover, for every $s \in \mathbb{R}^n$,

$$\begin{aligned} \widetilde{Q}(s) &= epi\,L(s,\cdot) \\ &:= \left\{\begin{array}{l} (v, v_0) \in \mathbb{R}^n \times \mathbb{R}: \\ v_0 \geq L(s,v) \end{array}\right\}. \end{aligned}$$

Then $L(s,\cdot)$ lsc $\forall s$.

On the other hand, $L(\cdot,\cdot)$ is Borel because $f_0(\cdot,\cdot)$ is lsc, and $f(\cdot,\cdot)$ is continuous, on $gph\,(U)$, which is closed by $(BH2)$.

As to $L^{**}(\cdot,\cdot)$, we have, due to $(BH2)$,

$$\begin{aligned} dom\, f_0^{**}(s,\cdot) &:= \{u \in \mathbb{R}^m : f_0^{**}(s,u) < \infty\} \\ &= co\,U(s). \end{aligned}$$

Therefore, by $(BH5)$ and $(BH6)$,

$$L^{**}: \mathbb{R}^n \times \mathbb{R}^n \to [0,\infty]$$
$$L^{**}(s,v) = \inf\left\{v_0 : \; (v, v_0) \in \widetilde{Q}_0(s)\right\},$$

where

$$\widetilde{Q}_0(s) := \begin{cases} \left\{\begin{array}{l} (v, v_0) \in \mathbb{R}^n \times \mathbb{R}: \\ v_0 \geq f_0^{**}(s,u), \\ v = f(s,u), \\ u \in co\,U(s) \end{array}\right\} \\ \text{if } s \in \Omega \\ \\ \emptyset \; \text{otherwise.} \end{cases}$$

$\square$

Define, for $s \in \Omega$, $u_0 \in \mathbb{R}^m$ and $\gamma \geq 0$,

$$R(s, \gamma u_0) := \left\{\begin{array}{l} u \in U(s): \\ B_0(s)(\gamma u_0 - u) = \\ = (1-\gamma) A_0(s) \end{array}\right\}$$

$$R_0(s, u_0) := \left\{\begin{array}{l} u \in co\,U(s): \\ B_0(s)(u_0 - u) = 0 \end{array}\right\}.$$

Proposition 8 *Assume the Basic Hypotheses* $(BH2)$ $-(BH6)$ *to hold true. Admit also validity of the following condition* (ACH)*:*

$\forall s \in \Omega \;\; \forall u_0 \in co\,U(s)$ *with*
$\inf\, f_0^{**}(s, R_0(s,u_0)) < \inf\, f_0(s, R(s,u_0))$

$\exists \lambda = \lambda(s,u_0) \in [0,1)$
$\exists \Lambda = \Lambda(s,u_0) \in (1,\infty)$ *for which*

$$\begin{aligned} & \inf\, f_0^{**}(s, R_0(s,u_0)) \\ = \; & (1-\alpha) \inf\, f_0(s, R(s, \lambda u_0)) + \\ & + \alpha \inf\, f_0(s, R(s, \Lambda u_0)) \end{aligned}$$

with $\alpha := \frac{1-\lambda}{\Lambda-\lambda}$ *and* $0 \cdot \infty := 0$.

Then $L(s,\cdot)$ (see (12)) is almost convex $\forall\, s$. Moreover, for any $\overline{x}_c(\cdot) \in W^{1,1}([a,b],\mathbb{R}^n)$ there exist measurable functions

$$\begin{aligned}&\lambda_1 = \lambda_1(\overline{x}_c, \overline{x}_c') : [a,b] \to [0,1]\\ &\lambda_2 = \lambda_2(\overline{x}_c, \overline{x}_c') : [a,b] \to [1,\infty)\\ &p : [a,b] \to [0,1]\end{aligned}$$

as in Proposition 2.

Proof: If $s \notin \Omega$ then

$$\begin{aligned}L^{**}(s,\cdot) &= L(s,\cdot)\\ &\equiv \infty .\end{aligned}$$

Therefore $L(s,\cdot)$ is almost convex.

Fix any $s \in \Omega$ and $\xi_0 \in \mathbb{R}^n$ for which

$$L^{**}(s,\xi_0) < L(s,\xi_0) . \tag{16}$$

Let $u_0 \in co\, U(s)$ be such that

$$\xi_0 = f(s,u_0) .$$

Since (see (15))

$$R_0(s,u_0) = H_0(s, f(s,u_0))$$

and (see (13)), for $\gamma \geq 0$,

$$\begin{aligned}&R(s,\gamma\, u_0)\\ =\;& H(s, f(s,\gamma\, u_0) + (\gamma-1)\, A_0(s)) ,\end{aligned}$$

by (16), (14) and (12),

$$\inf\, f_0^{**}(s, R_0(s,u_0)) < \inf\, f_0(s, R(s,u_0)) .$$

Hence there must

$$\begin{aligned}&\exists\, \lambda = \lambda(s,\xi_0) \in [0,1)\\ &\exists\, \Lambda = \Lambda(s,\xi_0) \in (1,\infty)\\ &\exists\, \alpha := \tfrac{1-\lambda}{\Lambda-\lambda} \in [0,1]\end{aligned}$$

for which

$$\begin{aligned}& L^{**}(s,\xi_0)\\ =\;& L^{**}(s, f(s,u_0))\\ =\;& \inf\, f_0^{**}(s, R_0(s,u_0))\\ =\;& (1-\alpha) \inf\, f_0(s, R(s,\lambda\, u_0)) +\\ & +\alpha\, \inf\, f_0(s, R(s,\Lambda\, u_0))\\ =\;& (1-\alpha)\, L(s,\lambda\, \xi_0) +\\ & +\alpha\, L(s,\Lambda\, \xi_0) .\end{aligned}$$

$\square$

4 Main results

In what follows, we will assume, to avoid trivialities, that

$$\inf_{(x(\cdot),u(\cdot)) \in \mathcal{Y}_{A,B}^{n,m}} J(x,u) < \infty . \tag{17}$$

Then our first main result is the following

Theorem 9 *Assume the Basic Hypotheses $(BH1)-(BH6)$ and condition (ACH). Suppose in addition that the set*

$$D_0 := \left\{ \begin{array}{l} (s, f_0(s,u) + \mu) :\\ (s,u) \in gph(U) ,\\ 0 = f(s,u) ,\;\; \mu \geq 0 \end{array} \right\}$$

is closed. Then existence of a relaxed solution implies existence of a true solution.

Proof: Let $(\overline{x}_c(\cdot), \overline{u}_c(\cdot)) \in \mathcal{Z}_{A,B}^{n,m}$ be a relaxed solution. Notice that, by (17),

$$\begin{aligned}&\textstyle\int_a^b L^{**}(\overline{x}_c(t), \overline{x}_c'(t))\, dt\\ \leq\;& \textstyle\int_a^b f_0^{**}(\overline{x}_c(t), \overline{u}_c(t))\, dt < \infty .\end{aligned} \tag{18}$$

Let us prove that $\overline{x}_c(\cdot)$ is a minimizer to the integral $I_c(\cdot)$.

Take $x(\cdot) \in \mathcal{X}_{A,B}^n$ for which

$$\int_a^b L^{**}\big(x(t), x'(t)\big)\, dt < \infty ,$$

so that

$$x'(t) \in f(x(t), co\, U(x(t))) \;\text{ a.e. on }\; [a,b] .$$

This implies, in particular, that

$$x(t) \in \Omega \;\; \forall\, t \in [a,b] .$$

Let $u : [a,b] \to \mathbb{R}^m$, be a measurable selection from the measurable and closed-valued multifunction (see [2])

$$\Gamma(t) := \left\{ \begin{array}{l} u \in \mathbb{R}^m :\\ u \in co\, U(x(t))\\ x'(t) \in f(x(t), u)\\ L^{**}(x(t), x'(t)) = f_0^{**}(x(t), u) \end{array} \right\} .$$

Then

$$(x(\cdot), u(\cdot)) \in \mathcal{Z}_{A,B}^{n,m}$$

and

$$\begin{aligned} & \int_a^b L^{**}\left(\overline{x}_c(t), \overline{x}_c'(t)\right) dt \\ \leq \; & \int_a^b f_0^{**}\left(\overline{x}_c(t), \overline{u}_c(t)\right) dt \\ \leq \; & \int_a^b f_0^{**}\left(x(t), u(t)\right) dt \\ = \; & \int_a^b L^{**}\left(x(t), x'(t)\right) dt, \end{aligned}$$

which proves the claim.

On the other hand, the function $L(\cdot, 0)$ is lsc, since

$$\begin{aligned} & epi\, L(\cdot, 0) \\ := \; & \left\{ \begin{array}{l} (s, s_0) \in \mathbb{R}^n \times \mathbb{R} : \\ s_0 \geq L(s, 0) \end{array} \right\} \\ = \; & D_0 \end{aligned}$$

is closed. Therefore, by Propositions 7 and 8, $L : \mathbb{R}^n \times \mathbb{R}^n \to [0, \infty]$ satisfies all the conditions of Proposition 3. Let $\overline{x}(\cdot)$ be a minimizer to the integral $I(\cdot)$. By (11) and (18),

$$\int_a^b L\left(\overline{x}(t), \overline{x}'(t)\right) dt < \infty .$$

Then, as above, it follows that there also exists a measurable $\overline{u} : [a, b] \to \mathbb{R}^m$ such that $(\overline{x}(\cdot), \overline{u}(\cdot)) \in \mathcal{Y}_{A,B}^{n,m}$ and, for every $(x(\cdot), u(\cdot)) \in \mathcal{Y}_{A,B}^{n,m}$

$$\begin{aligned} & \int_a^b f_0\left(\overline{x}(t), \overline{u}(t)\right) dt \\ = \; & \int_a^b L\left(\overline{x}(t), \overline{x}'(t)\right) dt \\ \leq \; & \int_a^b L\left(x(t), x'(t)\right) dt \\ \leq \; & \int_a^b f_0\left(x(t), u(t)\right) dt . \end{aligned}$$

This proves that $(\overline{x}(\cdot), \overline{u}(\cdot))$ is a true solution. □

Theorem 10 *Assume Hypotheses* $(BH1)$, $(BH3)$ $-(BH5)$ *and suppose* $U : \Omega \to 2^{\mathbb{R}^m} \setminus \emptyset$ *is an upper semicontinuous multifunction (see [2]) with compact values. In addition, admit validity of condition* (ACH).

Then, if

$$\begin{aligned} & \exists\, c_1 \in \mathbb{R} \;\; \exists\, c_2 > 0 \; : \\ & f_0(s, u) \geq c_1 + c_2\, |f(s, u)| \qquad (19) \\ & \forall\, (s, u) \in gph\,(U), \end{aligned}$$

there exists a true solution $(\overline{x}(\cdot), \overline{u}(\cdot))$. *Moreover,* $\overline{x}(\cdot)$ *is Lipschitz continuous.*

Proof: By (19) we may assume that Ω is compact. Then, since U is upper semicontinuous, Hypotheses $(BH2)$ and $(BH6)$ are satisfied. Moreover, $gph\,(U)$ and $gph\,(co\,U)$ are compact sets. Hence $L(\cdot, \cdot)$ is lsc and has superlinear growth at infinity. It follows, by Proposition 4, that there exists a Lipschitz continuous minimizer $\overline{x}(\cdot)$ to $I(\cdot)$. □

Theorem 11 *Assume* $(BH1) - (BH6)$ *and condition* (ACH). *Moreover, assume there exists a solution* $(\overline{x}_c(\cdot), \overline{u}_c(\cdot))$ *to* $(\mathcal{OCP})$ *such that, either the set*

$$F_0 := \{t \in [a, b] : \lambda_1(t) = 0\}$$

(see Proposition 8) has zero measure, or else

$$\overline{F_0} \setminus F_0 \text{ is a null set}$$

and $\exists\, t_c \in [a, b]$ *for which we can find some* $u_c \in U(\overline{x}_c(t_c))$ *satisfying :*

$$f(\overline{x}_c(t_c), u_c) = 0$$

and, for every $t \in [a, b]$,

$$\begin{aligned} & f_0(\overline{x}_c(t_c), u_c) \leq \\ & \leq \inf \left\{ \begin{array}{l} f_0(\overline{x}_c(t), u) : \\ u \in U(\overline{x}_c(t)) \\ f(\overline{x}_c(t), u) = 0 \end{array} \right\}. \end{aligned}$$

Then $(\mathcal{OCP})$ *has a solution.*

Proof: The function $L : \mathbb{R}^n \times \mathbb{R}^n \to [0, \infty]$ satisfies all the conditions of Proposition 5. □

5 Examples of application

Example 12 $(n = 1,\ m = 2)$ Let

- $\Omega = \mathbb{R}$
- $U : \Omega \to 2^{\mathbb{R}^2},\ U(s) \equiv U_0,$ with
  $U_0 := \;((\,(-\infty, -1] \cup [0, \infty)\,) \times \{0\}\,) \cup$
  $((\,(-\infty, 0] \cup [1, \infty)\,) \times \{-1\}\,)$
- $f_0 : gph\,(U) \to [0, \infty)$,
  $f_0(s, u) := g(s)\, h(u)$, where
  $g(s) := 1 + s^2$

  $$h(u) := \begin{cases} \sqrt{1 + |u|^2}, & |u| \geq 1 \\ \frac{1}{|u|} + \sqrt{6} - 1, & 0 < |u| < 1 \\ \sqrt{2}, & u = 0 ; \end{cases}$$

  set $f_0(s, u) := \infty$ for $(s, u) \notin gph\,(U)$

- $f: gph\,(co\,U) \to \mathbb{R}$,
  $f(s,u) = f(s,(u_1,u_2)) := u_1 + u_2$,
  so that $A_0(s) \equiv 0$ and $B_0(s) \equiv [1 \ \ 1]$.

Then, for $(a,A) = (0,1)$, $(b,B) = (2,0)$ problem $(OC\mathcal{P})$ has a solution $(x(\cdot), u(\cdot))$.

Indeed, with

$$\widetilde{x}(t) := \begin{cases} 1-t & \text{for } t \in [0,1] \\ 0 & \text{for } t \in [1,2] \end{cases}$$

$$\widetilde{u}(t) := \begin{cases} (-1,0) & \text{for } t \in [0,1] \\ (0,0) & \text{for } t \in (1,2] \end{cases}$$

we have

$$\begin{aligned} & \inf_{(x(\cdot),u(\cdot)) \in \mathcal{Y}^{n,m}_{A,B}} J(x,u) \\ \leq \ & J(\widetilde{x},\widetilde{u}) < \infty. \end{aligned}$$

Moreover the Basic Hypotheses $(BH1) - (BH6)$ hold true:

(BH1), (BH5): are imediate

(BH2.1): $gph\,(U) = \mathbb{R}\times U_0$ is closed

(BH2.2): $gph\,(co\,U) = \mathbb{R}\times[-1,0]$ is closed

(BH3): $epi\, f_0 =$

$$= \left\{ \begin{array}{l} ((s,u)\ ,a) \in gph\,(U)\times\mathbb{R} : \\ a \geq f_0(s,u) \end{array} \right\}$$

$$= \left\{ \begin{array}{l} ((s,u)\ ,a) \in \mathbb{R}\times U_0\times\mathbb{R} : \\ a \geq g(s)\,h(u) \end{array} \right\}$$

is a closed set, therefore $f_0(\cdot,\cdot)$ is lsc

(BH4): since $co\,U(s)$ is a closed set for every $s \in \mathbb{R}$,

$$\begin{aligned} dom\, f_0^{**} &:= \{(s,u) : f_0^{**}(s,u) < \infty\} \\ &= gph\,(co\,U); \end{aligned}$$

moreover,

$$f_0^{**}(s,u) = g(s)\,h^{**}(u),$$

hence, as above, $epi\, f_0^{**}$ is a closed set and $f_0^{**}(\cdot,\cdot)$ is lsc

(BH6): for every $s \in \mathbb{R}$, the sets

$$f(s, co\,U(s)) = \mathbb{R}$$

$$\left\{ \begin{array}{l} (f(s,u), f_0(s,u) + \mu) : \\ u \in U(s), \ \ \mu \geq 0 \end{array} \right\}$$

$$\begin{aligned} &= \left\{ \begin{array}{l} (v, v_0) \in \mathbb{R}\times\mathbb{R} : v_0 \geq f_0(s,u), \\ v = f(s,u), \ u \in U(s) \end{array} \right\} \\ &= \left((-\infty,-1] \times \left[g(s)\sqrt{2+(v+1)^2}, \infty \right) \right) \\ &\cup \left([0,1] \times \left[g(s)\sqrt{2+(v+1)^2}, \infty \right) \right) \\ &\cup \left([1,\infty) \times \left[g(s)\sqrt{1+v^2}, \infty \right) \right) \\ &\cup \left(\{0\} \times \left[g(s)\sqrt{2}, \infty \right) \right) \end{aligned}$$

and

$$\left\{ \begin{array}{l} (f(s,u), f_0^{**}(s,u) + \mu) : \\ u \in co\,U(s), \ \ \mu \geq 0 \end{array} \right\}$$

$$\begin{aligned} &= \left((-\infty,-1] \times \left[g(s)\sqrt{2+(v+1)^2}, \infty \right) \right) \\ &\cup \left([1,\infty) \times \left[g(s)\sqrt{1+v^2}, \infty \right) \right) \\ &\cup \left([-1,1] \times \left[g(s)\sqrt{2}, \infty \right) \right) \end{aligned}$$

are closed.

To verify validity of assumption (ACH), notice that for any $s \in \mathbb{R}$, $u_0 = (u_{01}, u_{02}) \in \mathbb{R}^m$ and $\gamma \geq 0$,

$$R(s, \gamma\, u_0) = \left\{ \begin{array}{l} (u_1,u_2) \in U_0 : \\ \gamma(u_{01} + u_{02}) = u_1 + u_2 \end{array} \right\}$$

$$R_0(s, u_0) = \left\{ \begin{array}{l} (u_1,u_2) \in \mathbb{R}^2 : \\ u_1 = u_{01} + u_{02} - u_2, \\ -1 \leq u_2 \leq 0 \end{array} \right\}.$$

Then, setting

$$C := \left\{ \begin{array}{l} u_0 \in co\,U(s) : \\ \inf\, f_0^{**}(s, R_0(s,u_0)) < \\ \inf\, f_0(s, R(s,u_0)) \end{array} \right\},$$

we obtain

$$C = C_1 \cup C_2$$

with

$$C_1 := \left\{ \begin{array}{l} (u_1,u_2) \in \mathbb{R}^2 : \\ -1 - u_2 < u_1 < -u_2, \ -1 \leq u_2 \leq 0 \end{array} \right\}$$

$$C_2 := \left\{ \begin{array}{l} (u_1,u_2) \in \mathbb{R}^2 : \\ -u_2 < u_1 < 1 - u_2, \ -1 \leq u_2 \leq 0 \end{array} \right\}.$$

For $u_0 \in C_1$

$$\exists\, \lambda = \lambda(s,u_0) \in [0,1)\,,\quad \lambda = 0$$
$$\exists\, \Lambda = \Lambda(s,u_0) \in (1,\infty)\,,\quad \Lambda = -\tfrac{1}{u_{01}+u_{02}}$$

for which

$$\begin{aligned} & \inf\, f_0^{**}(s, R_0(s,u_0)) = g(s)\sqrt{2} = \\ = \;& (1-\alpha)\; f_0(s,(0,0)) + \\ & +\alpha\, \inf\, f_0(s,(0,-1)) \\ = \;& (1-\alpha) \inf\, f_0(s, R(s,0)) + \\ & +\alpha\, \inf\, f_0\left(s, R\left(s, -\tfrac{u_{02}}{u_{01}+u_{02}}\right)\right) \\ = \;& (1-\alpha)\inf\, f_0(s, R(s,\lambda\, u_0)) + \\ & +\alpha\, \inf\, f_0(s, R(s, \Lambda\, u_0))\,. \end{aligned}$$

Similarly, for $u_0 \in C_2$

$$\exists\, \lambda = \lambda(s,u_0) \in [0,1)\,,\quad \lambda = 0$$
$$\exists\, \Lambda = \Lambda(s,u_0) \in (1,\infty)\,,\quad \Lambda = \tfrac{1}{u_{01}+u_{02}}$$

for which

$$\begin{aligned} & \inf\, f_0^{**}(s, R_0(s,u_0)) \\ = \;& (1-\alpha)\inf\, f_0(s, R(s,\lambda\, u_0)) + \\ & +\alpha\, \inf\, f_0(s, R(s, \Lambda\, u_0))\,. \end{aligned}$$

With regard to the existence of a relaxed solution, we can verify that the bipolar $L^{**} : \mathbb{R}^n \times \mathbb{R}^n \to [0,\infty)$,

$$L^{**}(s,v) = \begin{cases} g(s)\sqrt{2+(v+1)^2}, & v \geq 1 \\ g(s)\sqrt{2}, & 0 < v < 1 \\ g(s)\sqrt{1+v^2}, & u = 0 \end{cases}$$

satisfies all the hipotheses of [9, Theorem 2] (see [9, Example 4.2]). Then there exists a minimizer $\overline{x}_c(\cdot)$ to the integral $I_c(\cdot)$ and consequently also a solution $(\overline{x}_c(\cdot), \overline{u}_c(\cdot))$ to $(\mathcal{OCP}_c)$.

Finally, observe that the set

$$\begin{aligned} D_0 \;&:=\; \left\{ \begin{array}{l} (s, f_0(s,u)+\mu) : \\ (s,u) \in gph(U)\,, \\ 0 = f(s,u)\,,\;\; \mu \geq 0 \end{array} \right\} \\ &=\; \left\{ \begin{array}{l} (s\,, s_0) \in \mathbb{R}\times\mathbb{R} : s_0 \geq f_0(s,u)\,, \\ 0 = f(s,u)\,,\; s\in\mathbb{R},\; u \in U(s) \end{array} \right\} \\ &=\; \left\{ (s\,,s_0) \in \mathbb{R}\times\mathbb{R} : s_0 \geq g(s)\sqrt{2} \right\} \end{aligned}$$

is closed because $g(\cdot)$ is lsc. Thus, Theorem 9 ensures existence of a true solution.

Notice that in the previous example we could have avoided verify that the set D_0 is closed. Indeed, since there exists a minimizer $\overline{x}_c(\cdot)$ to the integral $I_c(\cdot)$, we can use Corollary 12 in [5] to obtain existence of another relaxed solution $(\widetilde{x}_c(\cdot), \widetilde{u}_c(\cdot))$ for which the set $F_0 := \{t \in [a,b] : \lambda_1(t) = 0\}$ has zero measure. Then all the hypotheses of Theorem 11 hold true.

Example 13 $(n = 1,\ m = 2)$ Let Ω, h and f as in Example 12. Assume

$$U : \Omega \to 2^{\mathbb{R}^2},$$
$$U(s) \equiv U_0 := U_{01} \times U_{02}\,,$$

with

$$U_{01} := \{0\} \cup [1,2]$$
$$U_{02} := \{-1, 0\}\,.$$

Let

$$f_0(s,u) := g(s)\, h(u)\,,$$

with

$$g : \mathbb{R} \to [1,\infty) \text{ lsc.}$$

Then, for every $(a,A), (b,B) \in \mathbb{R}^2$, $a \leq b$, problem $(OC\mathcal{P})$ has a solution $(x(\cdot), u(\cdot))$. Moreover, $x(\cdot)$ is Lipschitz continuous.

In fact, all the assumptions of Theorem 10 are satisfied. In particular, (19) holds true with $c_1 = 0$ and $c_2 = \frac{1}{2}$.

Another example to which Theorem 9 can be applied is the following (existence of a relaxed solution follows from Tonelli's Direct Method):

Example 14 $(n = 2,\ m = 2)$ Let

- $\Omega = \mathbb{R}^2$
- $U : \Omega \to 2^{\mathbb{R}^2},\ U(s) \equiv U_0 := \mathbb{R}^2$
- $f_0 : \mathbb{R}^4 \to [0,\infty)$,
  $f_0(s,u) := g(s) + h(u)$, where
  $g(s) := |s - s_0|^2$
  $$h(u) := \begin{cases} \left(|u|^2 - \gamma^2\right)^2, & u \neq 0 \\ 0, & u = 0 \end{cases}$$
- $f : \mathbb{R}^4 \to \mathbb{R}^2$,
  $f(s,(u_1,u_2)) := (u_1+u_2, u_1-u_2)\,.$

Then, for every s_0, A, $B \in \mathbb{R}^2$ and γ, a, $b \in \mathbb{R}$, $a \leq b$, problem $(OC\mathcal{P})$ has a solution $(x(\cdot), u(\cdot))$.

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