Fuzzy Causal Inferences based on Fuzzy Semantics of Fuzzy Concepts in Cognitive Computing

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Abstract: - Fuzzy semantics comprehension and fuzzy inference are two of the central abilities of human brains that play a crucial role in thinking, perception, and problem solving. A formal methodology for rigorously describing and manipulating fuzzy semantics and fuzzy concepts is sought for bridging the gap between humans and cognitive fuzzy systems. A mathematical model of fuzzy concepts is created based on concept algebra as the basic unit of fuzzy semantics for denoting languages entities in semantic analyses. The semantic operations of fuzzy modifiers and qualifiers on fuzzy concepts are introduced to deal with complex fuzzy concepts. On the basis the fuzzy semantic models, fuzzy causations and fuzzy causal inferences are formally elaborated by algebraic operations. The denotational mathematical structure of fuzzy semantics and fuzzy inferences not only explains the fuzzy nature of linguistic semantics and its comprehension, but also enables cognitive machines and fuzzy systems to mimic the human fuzzy inference mechanisms in cognitive informatics, cognitive linguistics, cognitive computing, and computational intelligence.

Key-Words: - Fuzzy systems, fuzzy semantics, fuzzy concept, fuzzy semantics, fuzzy inference, cognitive linguistics, cognitive informatics, cognitive computing, soft computing, computational intelligence.

1 Introduction

Fuzzy semantic comprehension and fuzzy inference are central abilities of human brains that play a crucial role in thinking, perception, and problem solving [1, 2, 4, 12, 13, 15, 18, 19, 24, 25, 26, 27, 29, 30, 33, 35, 36, 39, 40]. Semantics in linguistics represents the meaning or the intension and extension of a language entity [3, 7, 9, 10, 14]. Formal semantics [8, 9, 11, 20, 23, 28] focus on mathematical models for denoting meanings of symbols, concepts, functions, and behaviors, as well as their relations, which can be deduced onto a set of known concepts and behavioral processes in cognitive linguistics [5, 6, 29]. Causal inference is a cognitive process that deduces a proposition, particularly a causation, based on logical relations.

The taxonomy of semantics in natural languages can be classified into three categories [3, 9, 14, 29, 36] known as those of entities (noun and noun phrases), behaviors (verbs and verb phrases), and modifiers (adjectives, adverbs, and related phrases). Semantics can also be classified into the categories of *to-be, to-have,* and *to-do* semantics [28]. A *to-be* semantics infers the meaning of an equivalent relation between an unknown and a known entity or concept. A *to-have* semantics denotes the meaning of a possessive structure or a composite entity. A *to-do* semantics embodies the process of a behavior or an action in a 5-dimensional behavioral space [23, 28].

The fuzzy nature of linguistic semantics as well as its cognition stems from inherent semantic ambiguity, context variability, and individual perceptions influenced by heterogeneous knowledge bases. Almost all problems in natural language processing and semantic analyses are constrained by these fundamental issues. Lotfi A. Zadeh extended the methodologies for inferences by fuzzy sets and fuzzy logic [34, 37, 41], which provide a mathematical means for dealing with uncertainty and imprecision in reasoning, qualification, and quantification, particularly where vague linguistic variables are involved. Fuzzy inferences based on fuzzy sets are novel denotational mathematical means for rigorously dealing with degrees of matters, uncertainties, and vague semantics of linguistic entities, as well as for precisely reasoning the semantics of fuzzy causations. Typical fuzzy inference rules are those of fuzzy argument, implication, deduction, induction, abduction, and analogy [23, 27, 36, 41].

This paper presents a theory of fuzzy concepts and fuzzy semantics for formal semantic manipulation and fuzzy casual inference in cognitive systems and cognitive linguistics. The mathematical model of abstract fuzzy concepts is introduced in Section 2, which serves as the basic unit of fuzzy semantics in natural languages. A fuzzy concept is modeled as a fuzzy hyperstructure encompassing the fuzzy sets of attributes, objects, relations, and qualifications. Based on the mathematical model of fuzzy concepts, fuzzy semantic comprehension is reduced to a deduction process by algebraic operations on the fuzzy semantics. The mathematical model of fuzzy concept is extended to complex forms in Section 3 where fuzzy qualifiers are involved to modify fuzzy concepts. Fuzzy causal inference deduces a series of fuzzy semantics to determined implications. The denotational mathematical structures of fuzzy causal inferences are formally described in Section 4 by algebraic operations towards applications in cognitive linguistics, fuzzy systems, cognitive computing, and computational intelligence [6, 8, 12, 17, 21, 22].

2 The Fuzzy Discourse of Fuzzy Semantics and Fuzzy Inference

It is found that, although logical inferences may be carried out on the basis of classic sets and predicate logic, more inference mechanisms and rules such as those of intuitive, empirical, heuristic, perceptive, and semantic inferences, are fuzzy and uncertain [27, 31, 32, 33, 35, 38]. Lotfi A. Zadeh extends the classic set theory to fuzzy sets [34] and fuzzy logic [37]. Fuzzy set theory is one of the significant advances since George Boole's work on "*The Laws of Thought*" in 1854 and Cantor's classic set theory in 1874.

Fuzzy set theory is a suitable mathematical means for dealing with uncertainty and imprecision in reasoning, qualification, quantification, cognitive semantics, cognitive informatics, computational intelligence, cognitive computing, and denotational mathematics [13, 17, 19, 26, 30, 33].

Definition 1. The *discourse of fuzzy sets*, $\widetilde{\mathfrak{U}}_s$, is a universal enclosure of a finite set of *elements* $\mathcal{X}$ and a finite set of associated *weights of membership* $\mathfrak{W}$, i.e.:

$$\mathfrak{U}_s \triangleq \{(\mathcal{X}, \mathfrak{W}_{\mathcal{X}}) \mid \mathfrak{W}_{\mathcal{X}} : \mathcal{X} \rightarrow \mathbb{I} = [0,1]\} \qquad (1)$$

where $\mathbb{I}$ = [0, 1] is the *unit interval* in $\mathbb{R}$.

On the basis of the discourse of fuzzy sets as the universal context, a fuzzy set can be derived as follows.

Definition 2. A *fuzzy set* $\tilde{S}$ in $\widetilde{\mathfrak{U}}_s$ is a pair of a classic set *S* and a set of associate weights $\widetilde{W_S}$ of its elements $\tilde{e}$, known as the degrees of membership $\mu_{\tilde{S}}(\tilde{e})$, i.e.:

$$\begin{aligned}\tilde{S} &\triangleq \{(S, \widetilde{W_S})\}\\ &= \{(\tilde{e}, \widetilde{w_{\tilde{e}}}) \mid \tilde{e} \in \tilde{S} \subseteq \mathcal{X}, \widetilde{w_{\tilde{e}}} \in \widetilde{W_S} \subseteq \mathfrak{W}\}\\ &= \{(\tilde{e}, \mu_{\tilde{S}}(\tilde{e})) \mid \mu(\tilde{e}) = f : \tilde{e} \rightarrow \mathbb{I}',\ \mathbb{I}' = (0,1]\}\end{aligned} \qquad (2)$$

where $\widetilde{w_{\tilde{e}}}$ is determined by the *membership function* $\mu_{\tilde{S}}(\tilde{e})$ in the open unit interval $\mathbb{I}'$ = (0, 1], and $\mu_{\tilde{S}}(\tilde{e})$ is case dependent.

According to Definition 2, a fuzzy set can be perceived as a classic crisp set of elements plus an associated set of weights of the elements' memberships. From another perspective, a fuzzy set is perceived as a set of pairs of elements and weights of membership with respect to the fuzzy set. It is obvious that a fuzzy set is two-dimensional, while a crisp set is one-dimensional.

Example 1. An abstract fuzzy set $\widetilde{S_1}$ is as follows:

$$\begin{aligned}\widetilde{S_1} &= \{(\widetilde{e_i}, \mu_{\widetilde{S_1}}(\widetilde{e_i})) \mid 1 \leq i \leq 3\}\\ &= \{(\widetilde{e_1}, 0.6), (\widetilde{e_2}, 1.0), (\widetilde{e_3}, 0.2)\}\end{aligned}$$

It is noteworthy that, as expressed in Definition 2, a potential element $\tilde{e}_i$ in $\tilde{S}$ with zero membership is logically not considered as a member of $\tilde{S}$.

3 Fuzzy Semantics of Concepts in Fuzzy Inferences

The semantics of an entity in natural languages is used to be vaguely represented by a noun or noun phrase. In order to rigorously express the intension and extension of an entity expressed by a word, the noun entities can be formally described by an abstract concept in *concept algebra* [20] and *semantic algebra* [28]. An abstract concept is a cognitive unit to identify and model a concrete entity in the physical world or an abstract object in the perceived world, which can be formally described as follows.

Definition 3. Let $\mathfrak{O}$ denote a finite fuzzy set of *objects*, and $\mathfrak{A}$ be a finite fuzzy set of *attributes*. The *semantic discourse* of cognitive linguistics, $\mathfrak{U}_\Theta$, is a triple, i.e.:

$$\begin{aligned}\mathfrak{U}_\Theta &\triangleq (\mathfrak{O}, \mathfrak{A}, \mathfrak{R}) \\ &= \mathfrak{R}: \mathfrak{O}\to\mathfrak{O} \mid \mathfrak{O}\to\mathfrak{A} \mid \mathfrak{A}\to\mathfrak{O} \mid \mathfrak{A}\to\mathfrak{A}\end{aligned} \tag{3}$$

where $\mathfrak{R}$ is a fuzzy set of relations between $\mathfrak{O}$ and $\mathfrak{A}$.

On the basis of the *semantic discourse*, a formal fuzzy concept can be defined as a certain composition of subsets of the three kinds of elements known as the objects, attributes, and relations.

Definition 4. A *fuzzy concept* $\widetilde{C}$ in $\mathfrak{U}_\Theta$ is a hyperstructure of language entities denoted by a 5-tuple encompassing the fuzzy sets of attributes $\widetilde{A}$, objects $\widetilde{O}$, internal relations $\widetilde{R^i}$, external relations $\widetilde{R^o}$, and qualifications $\widetilde{Q}$, i.e.:

$$\widetilde{C} \triangleq (\widetilde{A}, \widetilde{O}, \widetilde{R^i}, \widetilde{R^o}, \widetilde{Q}) \tag{4}$$

where

- $\widetilde{A}$ is a fuzzy set of *attributes* as the *intension* of the concept $\widetilde{C}$:

$$\widetilde{A} = \{(\widetilde{a_1}, \mu_{\widetilde{A}}(\widetilde{a_1})), (\widetilde{a_2}, \mu_{\widetilde{A}}(\widetilde{a_2})), ..., (\widetilde{a_n}, \mu_{\widetilde{A}}(\widetilde{a_n}))\} \subseteq Þ\mathfrak{A} \tag{5}$$

where $Þ\mathfrak{A}$ denotes a power set of $\mathfrak{A}$.

- $\widetilde{O}$ is a fuzzy set of *objects* as the *extension* of the concept $\widetilde{C}$:

$$\widetilde{O} = \{(\widetilde{o_1}, \mu_{\widetilde{O}}(\widetilde{o_1})), (\widetilde{o_2}, \mu_{\widetilde{O}}(\widetilde{o_2})), ..., (\widetilde{o_m}, \mu_{\widetilde{O}}(\widetilde{o_m}))\} \subseteq Þ\mathfrak{O} \tag{6}$$

- $\widetilde{R^i}$ is a fuzzy set of *internal relations* between the fuzzy sets of objects $\widetilde{O}$ and attributes $\widetilde{A}$:

$$\begin{aligned}\widetilde{R^i} &= \widetilde{O} \times \widetilde{A} \subseteq Þ\mathfrak{R} \\ &= \mathop{R}_{j=1}^{\|\widetilde{O}\|} \mathop{R}_{i=1}^{\|\widetilde{A}\|} ((\widetilde{o_j}, \widetilde{a_i}), \mu_{\widetilde{O}}(\widetilde{o_i}) \bullet \mu_{\widetilde{A}}(\widetilde{a_j}))\end{aligned} \tag{7}$$

where the *big-R notation* [19, 26] expresses the Cartesian product of a series of repeated cross operations between o_j and a_i, $1 \le j \le m$ and $1 \le i \le n$, and $\|\widetilde{S}\|$ the number of elements in the fuzzy set.

- $\widetilde{R^o}$ is a fuzzy set of *external relations* between the fuzzy concept $\widetilde{C}$ and all potential ones $\widetilde{C'}$ in a knowledge base in $\mathfrak{U}_\Theta$:

$$\begin{aligned}\widetilde{R^o} &= \widetilde{C} \times \widetilde{C'} \subseteq Þ\mathfrak{R}, C' \ne C \wedge C' \sqsubseteq \mathfrak{U}_\Theta \\ &= \mathop{R}_{k=1}^{\|\mathfrak{O}\|} \{(\widetilde{C}, \widetilde{C'_k}), \mu_{\widetilde{R^o}}(\widetilde{R}_k) = \sigma)\}\end{aligned} \tag{8}$$

where $\widetilde{C'}$ is a fuzzy set of external concepts in $\mathfrak{U}_\Theta$, and the membership $\mu_{\widetilde{R^o}}(\widetilde{R}_k)$ is determined by the *conceptual equivalency* σ between the sets of fuzzy attributes from each fuzzy concepts, i.e.:

$$\sigma = \frac{\|\widetilde{A} \cap \widetilde{A'}\|}{\|\widetilde{A} \cup \widetilde{A'}\|} \tag{9}$$

- $\widetilde{Q}$ is a fuzzy set of *qualifications* that modifies the concept $\widetilde{C}$ by weights in (0, 1] as a special part of the external relations $\widetilde{R^o}$:

$$\widetilde{Q} = \{(q_1, \omega(q_1)), (q_2, \omega(q_2)), ..., (q_p, \omega(q_p)))\} \subseteq Þ\mathfrak{R} \tag{10}$$

where $\widetilde{Q}$ is initially empty when the concept is created as an independent one. However, it obtains qualified properties and weights when the fuzzy concept is modified by an adjective or adjective phrase, or it is comparatively evaluated with other fuzzy concepts.

In the fuzzy concept model, Eqs. 7 and 8 denote general internal and external relations, respectively. A concrete fuzzy relation in a specific fuzzy concept will be an instantiation of the general relations tailored by a given characteristic matrix on the Cartesian products.

As described in Definition 4, the important properties of a formal fuzzy concept are the fuzzy set of essential attributes as its *intension*; the fuzzy set of instantiated objects as its *extension*; and the adaptive capability to autonomously interrelate the concept to other concepts in an existing knowledge base in $\mathfrak{U}_\Theta$.

Example 2. A fuzzy concept 'pen', $\widetilde{C}(pen)$, can be formally described according to Definition 4 as follows:

$$
\begin{aligned}
\widetilde{C}(pen) &\triangleq \widetilde{C}(\widetilde{A},\widetilde{O},\widetilde{R^i},\widetilde{R^o},\widetilde{Q}) \\
&= \widetilde{pen}((\widetilde{A},\widetilde{\mu_C(A)}),(\widetilde{O},\widetilde{\mu_C(O)}),\widetilde{R^{i'}},\widetilde{R^{o'}},\widetilde{Q}) \\
&= \begin{cases}
\widetilde{A} = \{(a_1,\mu(a_1)),(a_2,\mu(a_2)),(a_3,\mu(a_3)),(a_4,\mu(a_4)\} \\
\quad = \{(writing\_tool,\ 1.0),\ (ink,\ 0.9),\ (nib,\ 0.9), \\
\qquad (ink\_container,\ 0.8)\} \\
\widetilde{O} = \{(o_1,\mu(o_1)),(o_2,\mu(o_2)),(o_3,\mu(o_3)),(o_4,\mu(o_4)\} \\
\quad = \{(ballpoint,\ 1.0),\ (fountain,\ 1.0), \\
\qquad (pencil,\ 0.9),\ (brush,\ 0.7)\} \\
\widetilde{R^i} = \widetilde{O} \times \widetilde{A} \\
\widetilde{R^o} = \widetilde{C} \times \widetilde{C}' \\
\widetilde{Q} = \varnothing
\end{cases}
\end{aligned}
$$

Example 3. A fuzzy concept 'man', $\widetilde{C}(man)$, can be formally described based on Definition 4 as follows:

$$
\begin{aligned}
\widetilde{C}(man) &\triangleq \widetilde{C}(\widetilde{A},\widetilde{O},\widetilde{R^i},\widetilde{R^o},\widetilde{Q}) \\
&= \widetilde{man}((\widetilde{A},\widetilde{\mu_C(A)}),(\widetilde{O},\widetilde{\mu_C(O)}),\widetilde{R^{i'}},\widetilde{R^{o'}},\widetilde{Q}) \\
&= \begin{cases}
\widetilde{A} = \{(human\_being,\ 1.0),\ (male,\ 1.0),\ (adult,\ 0.9)\} \\
\widetilde{O} = \{(American,\ 1.0),\ (Australia,\ 1.0), \\
\qquad (business\_man,\ 1.0),\ ...\} \\
\widetilde{R^{i'}} = \widetilde{O} \times \widetilde{A} \\
\widetilde{R^{o'}} = \widetilde{C} \times \widetilde{C}' \\
\widetilde{Q} = \varnothing
\end{cases}
\end{aligned}
$$

Applying the fuzzy concept model as a basic unit of semantic knowledge in $\mathfrak{U}_\Theta$, the fuzzy semantics in natural languages can be expressed as a mapping from a fuzzy language entity to a determined fuzzy concept where its sets of fuzzy attributes, objects, relations, and qualifications are specified.

Definition 5. The *fuzzy semantics of an entity* $\widetilde{e}$, $\widetilde{\Theta}(\widetilde{e})$, is an equivalent fuzzy concept $\widetilde{C}_e$ in $\mathfrak{U}_\Theta$, i.e.:

$$
\begin{aligned}
\widetilde{\Theta}(\widetilde{e}) &\triangleq \widetilde{\Theta}(\widetilde{e} = \widetilde{C_e}) \\
&= \widetilde{C}_e(\widetilde{A_e},\widetilde{O_e},\widetilde{R_e^i},\widetilde{R_e^o},\widetilde{Q_e})
\end{aligned} \tag{11}
$$

where $\widetilde{C}_e$ is denoted according to the generic model of fuzzy concepts as given in Definition 4.

Example 4. The fuzzy semantics of a language entity 'pen', denoted by $\Theta(\widetilde{C}(pen))$, can be formally derived according to Definition 5 and Example 2 as follows:

$$
\begin{aligned}
\widetilde{\Theta_e}(pen) &\triangleq \widetilde{\Theta}(\widetilde{e} = \widetilde{C}(pen)) \\
&= \widetilde{pen}((\widetilde{A},\widetilde{\mu_{\widetilde{A}}(A)}),(\widetilde{O},\widetilde{\mu_{\widetilde{O}}(O)}),\widetilde{R^i},\widetilde{R^o},\widetilde{Q}) \\
&= \begin{cases}
\widetilde{A} = \{(writing\_tool,\ 1.0),\ (ink,\ 0.9),\ (nib,\ 0.9), \\
\qquad (ink\_container,\ 0.8)\} \\
\widetilde{O} = \{(ballpoint,\ 1.0),\ (fountain,\ 1.0), \\
\qquad (pencil,\ 0.9),\ (brush,\ 0.7)\} \\
\widetilde{R^i} = \widetilde{O} \times \widetilde{A} \\
\widetilde{R^o} = \widetilde{C} \times \widetilde{C}' \\
\widetilde{Q} = \varnothing
\end{cases}
\end{aligned}
$$

Example 5. Similarly, the fuzzy semantics of a language entity 'man', denoted by $\Theta(\widetilde{C}(man))$, can be formally derived based on Definition 5 and Example 3 as follows:

$$
\begin{aligned}
\widetilde{\Theta_e}(man) &\triangleq \widetilde{\Theta}(\widetilde{e} = \widetilde{C}(man)) \\
&= \widetilde{man}((\widetilde{A},\widetilde{\mu_{\widetilde{A}}(A)}),(\widetilde{O},\widetilde{\mu_{\widetilde{O}}(O)}),\widetilde{R^{i'}},\widetilde{R^{o'}},\widetilde{Q}) \\
&= \begin{cases}
\widetilde{A} = \{(human\_being,\ 1.0),\ (male,\ 1.0), \\
\qquad (adult,\ 0.9)\} \\
\widetilde{O} = \{(American,\ 1.0),\ (Australia,\ 1.0), \\
\qquad (business\_man,\ 1.0),\ ...\} \\
\widetilde{R^{i'}} = \widetilde{O} \times \widetilde{A} \\
\widetilde{R^{o'}} = \widetilde{C} \times \widetilde{C}' \\
\widetilde{Q} = \varnothing
\end{cases}
\end{aligned}
$$

Therefore, on the basis of the formal fuzzy concept model (Definition 4) and fuzzy semantic model (Definition 5), fuzzy semantic analyses and comprehension in natural languages can be formally described as a deductive process from a fuzzy entity to a determined fuzzy concept.

Corollary 1. The *rule of semantic deduction* states that the semantics of a given fuzzy entity is comprehended in semantic analysis, *iff* its fuzzy semantics is reduced onto a known fuzzy concept with determined membership and weight values.

4 Fuzzy Semantics of Modifiers on Concepts in Fuzzy Inferences

The semantics of fuzzy concepts is usually modified by an adjective or an adjective phrase in language expressions in order to fine tune its qualification such as degree, scope, quality, constraint, purpose, and etc. Therefore, the fuzzy semantics of fuzzy concepts as developed in Section 3 can be extended to deal with composite semantics of noun phrases modified by determiners and degree words [20, 28, 29, 35, 36, 38].

The modifier in cognitive linguistics is words or phrases that elaborate, limit, and qualify a noun or noun phrase in the categories of determiners, qualifiers, degrees, and negations [6, 29]. A fuzzy modifier can be represented as a fuzzy set with certain weights of memberships [35, 36, 38]. For instance, Zadeh considered the fuzzy effects of some special adverbs on adjectives such as 'very, very', 'very little', 'positive', and 'negative' in 1975, which were modeled as nonlinear exponential weights on the target adjectives [34]. However, the general semantics relations between a fuzzy linguistic entity (noun) and its fuzzy modifier (adverb-adjective phrase) are yet to be studied.

Definition 6. A *fuzzy modifier* $\tilde{\tau}$ is a special fuzzy set that represents an adjective or adjective phrase in natural languages where its memberships are replaced by intentional weights of the modifier, $\omega_{\tilde{\tau}}(\tau_k)$, $1 \le k \le z$, and z is a constant, i.e.:

$$\begin{aligned}\tilde{\tau} &\triangleq \{\mathop{R}_{k=1}^{z}(\tau_k, \omega_{\tilde{\tau}}(\tau_k))\},\ \omega_{\tilde{\tau}}(\tau_k) \in (0,1] \\ &= \{(\tau_1, \omega(\tau_1)), (\tau_2, \omega(\tau_2)), ..., (\tau_z, \omega(\tau_z))\}\end{aligned} \tag{12}$$

where the weights of $\tilde{\tau}$ is normalized in the open unit interval $\mathbb{I}$'= (0, 1].

Example 6. A fuzzy modifier 'good' on the *quality* of a fuzzy entity can be formally described as a fuzzy set $\tilde{\tau}(good)$ according to Definition 6 as follows:

$$\begin{aligned}\tilde{\tau}(good) &= \{\mathop{R}_{k=1}^{4}(\tau_k(quality), \mu_{\tilde{\tau}}(\tau_k))\} \\ &= \{(neutral,\ 0.1),\ (ok,\ 0.6), \\ &\quad (excellent,\ 0.8),\ (perfect,\ 1.0)\}\end{aligned} \tag{13}$$

Example 7. A fuzzy modifier 'old' on the fuzzy entity *ages* can be formally described as a fuzzy set $\tilde{\tau}(old)$ as follows:

$$\begin{aligned}\tilde{\tau}(old) &= \{\mathop{R}_{k=1}^{6}(\tau_k(age), \omega_{\tilde{\tau}}(\tau_k))\} \\ &= \{([1\text{-}20], 0), ([21\text{-}30], 0.1), ([31\text{-}50], 0.4), \\ &\quad ([51\text{-}65], 0.7), ([66\text{-}80], 0.9), ([>80], 1.0)\}\end{aligned} \tag{14}$$

Definition 7. A *fuzzy qualifier* $\tilde{\delta}$ is a *special* fuzzy set of degree adverbs or adverb phrases to modify $\tilde{\tau}$ in natural languages where their memberships are replaced by *intentional weights* of degree and extends, $\omega_{\tilde{\delta}_l}(\delta_l)$, $1 \le l \le q$, and q is a constant, i.e.:

$$\begin{aligned}\tilde{\delta} &\triangleq \{\mathop{R}_{l=1}^{p}(\delta_l, \omega_{\tilde{\delta}}(\delta_l))\},\ \omega_{\tilde{\delta}}(\delta_l) \in \pm(0,3] \\ &= \{(\delta_1, \omega(\tau_1)), (\delta_2, \omega(\delta_2)), ..., (\delta_p, \omega(\delta_p))\}\end{aligned} \tag{15}$$

where the weights of $\tilde{\delta}$ is constrained in the domain $\pm[1,3]$ corresponding to the *neutral* (1), *comparative* (2), and *superlative* (3) degrees of adverbs and adjectives in natural languages.

Example 8. A typical fuzzy set of qualifiers, $\tilde{\delta}$, can be described according to Definition 7 as follows:

$$\begin{aligned}\tilde{\delta} = \{&(definitely\_not,\ -3.0),\ (imperfectly,\ -1.5), \\ &(neutral\_negative,\ -1.0),\ (somewhat,\ 0.5), \\ &(fairly,\ 1.2),\ (quite,\ 1.5),\ (excellently,\ 2.0), \\ &(extremely,\ 3.0)\},\ \omega_{\tilde{\delta}}(\delta) \in \pm(0,3]\end{aligned} \tag{16}$$

Definition 8. A *composite fuzzy modifier* $\widetilde{\delta \bullet \tau}$ is a product of a fuzzy qualifier $\tilde{\delta}$ and a fuzzy modifier $\tilde{\tau}$. The value of the composite modifiers is determined by the product of their weights, i.e.:

$$\begin{aligned}\Theta(\tilde{\delta} \bullet \tilde{\tau}) &\triangleq \Theta(\widetilde{\delta(y) \bullet \tau(x)}) \\ &= \omega_{\tilde{\delta}}(y) \bullet \omega_{\tilde{\tau}}(x),\ 0 < \omega_{\tilde{\tau}}(x) \le 1, \\ &\quad -3 \le \omega_{\tilde{\delta}}(y) \le 3,\ \omega_{\tilde{\delta}}(y) \ne 0\end{aligned} \tag{17}$$

where the *combined domain* of composite modifiers is $\pm(0,3]$ in order to be consistent with the modifiers in real-world languages.

In case a weight of the fuzzy qualifiers is less than zero, the composite modifier represents a negative intention. For instance, $\tilde{\delta} \bullet \tilde{\tau} = \tilde{\delta}(neutral\_negative) \bullet \tilde{\tau}(good)$ implies a weight of qualification in the semantics as $\omega(\tilde{\delta}(neutral\_negative)) \bullet \omega(\tilde{\tau}(good)) = -1 \bullet 0.6 = -0.6.$

On the basis of the formal semantics of fuzzy modifiers $\tilde{\tau}$, qualifiers $\tilde{\delta}$, and composite modifiers $\widetilde{\tau'} = \widetilde{\delta\tau}$, the composite fuzzy semantics of language entities modified by $\widetilde{\delta\tau}$ can be quantitatively expressed.

Definition 9. The *composite fuzzy semantic* of a fuzzy concept $\widetilde{C}$ qualified by a *fuzzy* modifier $\tilde{\tau}$, qualifier $\tilde{\delta}$, and/or a composite fuzzy modifier $\widetilde{\tau'} = \widetilde{\delta\tau}$, denoted by $\widetilde{\Theta}(\widetilde{C'}) = \widetilde{\Theta}(\widetilde{\tau'} \bullet \widetilde{C})$, is a complex semantics of the fuzzy concept $\widetilde{C}$ qualified by a certain weight of the composite modifier, i.e.:

$$
\begin{aligned}
\widetilde{\Theta}(\widetilde{\tau'} \bullet \widetilde{C}) &\triangleq \widetilde{\Theta}(\widetilde{C} \mid \widetilde{Q'} = \widetilde{\delta\tau}) \\
&= \widetilde{C}(\widetilde{A}, \widetilde{O}, \widetilde{R^i}, \widetilde{R^o}, \widetilde{Q'} \mid \widetilde{Q'} = \widetilde{\delta\tau}) \\
&= \widetilde{C}((\widetilde{A}, \widetilde{\mu_{\tilde{A}}(A)}), (\widetilde{O}, \widetilde{\mu_{\tilde{O}}(O)}), \widetilde{R^i}, \widetilde{R^o}, (\widetilde{Q'} = \widetilde{\delta\tau})) \\
&= \widetilde{C}(\widetilde{A}, \widetilde{O}, \widetilde{R^i}, \widetilde{R^o}, \widetilde{Q'})
\end{aligned} \tag{18}
$$

where the fuzzy set of composite modifiers imposes a specific set of weights of intentional qualifications $\widetilde{Q'}$ in the modified semantics of the target fuzzy concept, and $\tilde{\delta} = 1$ if it is absent.

Example 9. Given a fuzzy concept $\widetilde{C}(pen)$ as obtained in Example 2, the composite semantics $\widetilde{\Theta}(\tilde{\tau} \bullet \widetilde{C}) = \widetilde{\Theta}(excellent\_pen)$ qualified by the fuzzy modifier $\tilde{\tau}(good)$ can be determined according to Definition 9, i.e.:

$$
\begin{aligned}
&\widetilde{\Theta}(\widetilde{\delta\tau} \bullet \widetilde{C}) = \widetilde{\Theta}(\tilde{\tau}(good) \bullet \widetilde{pen}) \\
&= \widetilde{excellent\_pen}((\widetilde{A}, \widetilde{\mu_{\tilde{A}}(A)}), (\widetilde{O}, \widetilde{\mu_{\tilde{O}}(O)}), \widetilde{R^i}, \widetilde{R^o}, \\
&\qquad (\widetilde{Q'} = \tau_0 \mid \tau_0 = \tau(excellent) = 0.8)) \\
&= \begin{cases}
\widetilde{A} = \{(writing,\ 1.0),\ (ink,\ 0.9),\ (nib,\ 0.9), \\
\qquad (ink\_container,\ 0.8)\} \\
\widetilde{O} = \{(ballpoint,\ 1.0),\ (fountain,\ 1.0),\ (pencil,\ 0.9), \\
\qquad (brush,\ 0.7)\} \\
\widetilde{R^i} = \widetilde{O} \times \widetilde{A} \\
\widetilde{R^o} = \widetilde{C} \times \widetilde{C'} \\
\widetilde{Q'}(excellent) = 0.8
\end{cases}
\end{aligned}
$$

Example 10. The fuzzy semantics of $\widetilde{C}(excellent\_pen)$ obtained in Example 9 may be further modified by a qualifier $\tilde{\delta}(extremely)$ that results in $\widetilde{C}(extremely\_excellent\_pen)$ as follows:

$$
\begin{aligned}
&\widetilde{\Theta}(\widetilde{\delta\tau} \bullet \widetilde{C}) = \widetilde{\Theta}(\tilde{\delta}(extremely) \bullet \tilde{\tau}(good) \bullet \widetilde{pen}) \\
&= \widetilde{extremely\_excellent\_pen}((\widetilde{A}, \widetilde{\mu_{\tilde{A}}(A)}), (\widetilde{O}, \widetilde{\mu_{\tilde{O}}(O)}), \widetilde{R^i}, \widetilde{R^o}, \\
&\qquad (\widetilde{Q'} = \tau_0' \mid \tau_0' = \tilde{\delta}(extremely) \bullet \tau(excellent) = 3.0 \bullet 0.8)) \\
&= \begin{cases}
\widetilde{A} = \{(writing\_tool,\ 1.0),\ (ink,\ 0.9),\ (nib,\ 0.9), \\
\qquad (ink\_container,\ 0.8)\} \\
\widetilde{O} = \{(ballpo\mathrm{int},\ 1.0),\ (fountain,\ 1.0),\ (pencil,\ 0.9), \\
\qquad (brush,\ 0.7)\} \\
\widetilde{R^i} = \widetilde{O} \times \widetilde{A} \\
\widetilde{R^o} = \widetilde{C} \times \widetilde{C'} \\
\widetilde{Q'}(extremely\_excellent) = 2.4
\end{cases}
\end{aligned}
$$

Example 11. Given a fuzzy concept $\widetilde{C}(man)$ as described in Example 3, the composite semantics $\widetilde{\Theta}(\tilde{\tau} \bullet \widetilde{C}) = \widetilde{\Theta}(old\_man)$ qualified by the fuzzy modifier $\tilde{\tau}$(*old*) can be determined according to Definition 9 as follows:

$$
\begin{aligned}
\widetilde{\Theta}(\tilde{\tau} \bullet \widetilde{C}) &= \widetilde{\Theta}(\tilde{\tau}(old) \bullet \widetilde{man}),\ \tau_0 = \tau(60) = 0.7 \\
&= \widetilde{old\_man}((\widetilde{A}, \widetilde{\mu_{\tilde{A}}(A)}), (\widetilde{O}, \widetilde{\mu_{\tilde{O}}(O)}), \widetilde{R^i}, \widetilde{R^o}, (\widetilde{Q} = \tau_0)) \\
&= \begin{cases}
\widetilde{A} = \{(human\_being,\ 1.0),\ (male,\ 1.0), \\
\qquad (adult,\ 0.9)\} \\
\widetilde{O} = \{(American,\ 1.0),\ (Australia,\ 1.0), \\
\qquad (business\_man,\ 1.9),\ ...\} \\
\widetilde{R^i} = \widetilde{O} \times \widetilde{A} \\
\widetilde{R^o} = \widetilde{old\_man} \times \widetilde{C'} \\
\widetilde{Q}(old) = 0.7
\end{cases}
\end{aligned}
$$

Example 11 indicates that, against the fuzzy qualifier $\tilde{\tau}(old)$, a man in the age of 60 is 0.7 (*quite likely*) as an old man. Similarly, other instantiations modified by the qualifiers may denote that a man in the age of 25 is 0.1 (*unlikely*) as an old man; and a man in the age of 85 is 1.0 (*definitely*) as an old man.

Example 12. The fuzzy semantics of $\widetilde{C}(old\_man)$ obtained in Example 11 may be further modified by a qualifier $\tilde{\delta}(quite)$ that results in $\tilde{C}(a\_quite\_old\_man)$ as follows:

$$\begin{aligned}
\tilde{\Theta}(\tilde{\delta\tau} \bullet \tilde{C}) &= \tilde{\Theta}(\tilde{\delta}(quite) \bullet \tilde{\tau}(old) \bullet \widetilde{man}) \\
&= \widetilde{a_quite_old_man}((\tilde{A}, \widetilde{\mu_{\tilde{A}}}(A), (\tilde{O}, \widetilde{\mu_{\tilde{O}}}(O), \widetilde{R^i}, \widetilde{R^o}, \\
&\qquad (\tilde{Q}' = \tau_0' \mid \tau_0' = \tilde{\delta}(quite) \bullet \tau(60) = 1.5 \bullet 0.7))
\end{aligned}$$

$$= \begin{cases}
\tilde{A} = \{(human_being,\ 1.0),\ (male,\ 1.0),\ (adult,\ 0.9)\} \\
\tilde{O} = \{(American,\ 1.0),\ (Australia,\ 1.0), \\
\qquad (business_man,\ 1.0),\ ...\} \\
\widetilde{R^i} = \tilde{O} \times \tilde{A} \\
\widetilde{R^o} = \widetilde{old_man} \times \tilde{C}' \\
\tilde{Q}'(quite_old) = 1.05
\end{cases}$$

The fuzzy nature of language semantics and their comprehension is formally explained by the mathematical models of fuzzy concepts and fuzzy semantics qualified by fuzzy modifiers.

5 Fuzzy Causal Inferences

Logical causations are a semantic relation between serial events, states, phenomena, and behaviors. Inference is a cognitive process that reasons a possible causation from given premises between a pair of cause and effect where reasoning is a synonym of inference as an action to think, understand, and form judgments logically. Causal inference is one of the central capabilities of the human brain that plays a crucial role in thinking, perception, reasoning, and problem solving [2, 15, 22, 31]. A causal inference can be conducted based on empirical observations, formal reasoning, and statistical norms [1, 24, 27, 31].

5.1 Fuzzy Causations and Fuzzy Inferences

On the basis of fuzzy set theory and fuzzy logic [32, 35], traditional logical inferences can be extended to fuzzy inferences where the premises of reasoning become fuzzy expressions in fuzzy logic rather than those of Boolean expressions in traditional logical inferences.

Definition 10. The *discourse of fuzzy causality* $\tilde{\mathfrak{U}}_c$ is a triple:

$$\tilde{\mathfrak{U}}_c \triangleq (\tilde{\Xi}, \tilde{\mathfrak{E}}, \tilde{\mathfrak{R}}) \quad (19)$$

where $\tilde{\Xi}$ denotes a finite set of fuzzy states (Definition 11), $\tilde{\mathfrak{E}}$ a finite set of fuzzy events (Definition 12), and $\tilde{\mathfrak{R}}$ a finite set of fuzzy relations (Definition 13).

Definition 11. The *fuzzy set of fuzzy states* $\tilde{\Xi}$ in $\tilde{\mathfrak{U}}_c$ is a set of dynamic states $\tilde{s}_i \in \tilde{\Xi}$ of entities and/or causations constrained by associated degrees of membership $\mu_{\tilde{\Xi}}(\tilde{s}_i)$, i.e.:

$$\tilde{\Xi} \triangleq \{(\tilde{s}_i, \mu_{\tilde{\Xi}}(\tilde{s}_i)) \mid \tilde{s}_i \in \tilde{\Xi} \wedge \mu_{\tilde{\Xi}}(\tilde{s}_i) \in \mathbb{I}' = (0,1]\} \quad (20)$$

Definition 12. The *fuzzy set of fuzzy events* $\tilde{\mathfrak{E}}$ in $\tilde{\mathfrak{U}}_c$ is a fuzzy set of changes of states in $\tilde{\Xi}$, i.e.:

$$\begin{aligned}
\tilde{\mathfrak{E}} &\triangleq \frac{d\tilde{\Xi}}{dt} = \frac{\tilde{\Xi}_t \oplus \tilde{\Xi}_{t'}}{t-t'}\Big|_{t-t'=1} = \tilde{\Xi}_t \oplus \tilde{\Xi}_{t'} \\
&= \{(\tilde{e}_i, \mu_{\tilde{\mathfrak{E}}}(\tilde{e}_i)) \mid \tilde{e}_i \in \tilde{\mathfrak{E}} \subseteq \tilde{\Xi}_t \oplus \tilde{\Xi}_{t'} \subseteq \tilde{\Xi}, \\
&\qquad \mu_{\tilde{\mathfrak{E}}}(\tilde{e}_i) \in \mathbb{I}' = (0,1]\}
\end{aligned} \quad (21)$$

Definition 13. The *fuzzy set of relations* $\tilde{\mathfrak{R}}$ in $\tilde{\mathfrak{U}}_c$ is a Cartesian product between the fuzzy sets of states and/or events, i.e.:

$$\tilde{\mathfrak{R}} = \tilde{\Xi} \times \tilde{\Xi} \mid \tilde{\mathfrak{E}} \times \tilde{\mathfrak{B}} \quad (22)$$

On the basis of the fuzzy causal discourse $\mathfrak{U}_c$, a fuzzy *causation* is a fuzzy relation of a logical consequence between a sole or multiple fuzzy causes and a single or multiple fuzzy effects.

Definition 14. A *fuzzy causation* $\tilde{\xi}$, $\tilde{\xi} \in \tilde{\mathfrak{R}}$, in $\tilde{\mathfrak{U}}_c$ is a relation that maps a nonempty fuzzy set of causes $\tilde{\mathcal{C}}$ into a nonempty fuzzy set of effects $\tilde{\mathcal{E}}$, i.e.:

$$\xi \triangleq f_\xi : \tilde{\mathcal{C}} \to \tilde{\mathcal{E}},\ \xi \in \tilde{\mathfrak{R}}, \tilde{\mathcal{C}} \subseteq \tilde{\Xi} \cup \tilde{\mathfrak{E}}, \tilde{\mathcal{E}} \subseteq \tilde{\Xi} \quad (23)$$

The fuzzy causal relations may be 1-1, 1-*n*, *n*-1, and *n*-*m*, where *n* and *m* are integers greater than one that represent multiple relations. The fuzzy *cause* ($\tilde{\mathcal{C}}$) in a fuzzy causation in $\tilde{\mathfrak{U}}_c$ is a premise state such as an event, phenomenon, action, behavior, or existence. Equivalent to a fuzzy cause, a fuzzy *reason* is a premise of an argument in support of a belief or causation. However, the fuzzy *effect* ($\tilde{\mathcal{E}}$) in a fuzzy causation in $\tilde{\mathfrak{U}}_c$ is a consequent or conclusive state such as an event, phenomenon, action, behavior, or existence.

Based on the formal models of fuzzy semantics, fuzzy inferences and fuzzy causal analyses can be rigorously manipulated by humans as well as fuzzy cognitive systems.

5.2 Rules of Fuzzy Causal Inferences

Fuzzy causal inference is a denotational mathematical methodology for rigorously dealing with degrees of matters, uncertainties, and vague semantics of linguistic variables, as well as for precisely reasoning the semantics of fuzzy causations. Fuzzy causal inference enables qualitative and/or quantitative evaluation of the degree of a given causation on the basis of fuzzy expressions [27, 32, 33, 34, 35, 36]. The fuzzy expressions in fuzzy inferences are a qualification or quantification of linguistic variables [37, 38] formalized by fuzzy concepts, fuzzy semantics, and fuzzy causations as described in preceding sections.

According to Definitions 10, 13, and 14, there are two types of fuzzy causal inferences known as the *fuzzy relational inference* and *fuzzy behavioral inference*. The former is a "*to be*" type fuzzy causal relation between a pair of fuzzy cause and effect; while the latter is a "*to-do*" type fuzzy causal relation between a pair of fuzzy event and fuzzy behavior.

Definition 15. A *fuzzy relational causal inference*, $\tilde{\kappa}$, is a process that deduces a valid causation $\widetilde{C'} \vdash \tilde{e}$ between a set of fuzzy causes $\widetilde{C'} \subset \tilde{\mathcal{C}}$ and a fuzzy effect $\tilde{e} \in \tilde{\mathcal{E}}$ by qualitative causal differential on a set of potential fuzzy causes $\tilde{C} = \mathop{R}_{i=1}^{n}(\tilde{c}_i, \mu_{\tilde{C}}(\tilde{c}_i)), \tilde{C} \supseteq \widetilde{C'}$ with respect to $\tilde{c}_i$, $1 \leq i \leq n' = \|\widetilde{C'}\| \leq n = \|\tilde{C}\|$, i.e.:

$$\begin{aligned}
\tilde{\kappa} &\triangleq \{\mathop{R}_{i=1}^{n} \frac{\partial}{\partial \tilde{c}_i}\tilde{C} \to \tilde{e}\} \vdash \tilde{e} \\
&= \{\mathop{R}_{i=1}^{n} (\tilde{c}_i \mid \frac{\partial}{\partial c_i}(\mu_{\tilde{C}}(\tilde{c}_i) \sim \tau)) \to \tilde{e})\} \vdash \tilde{e} \\
&= \{\mathop{R}_{i=1}^{n} [\tilde{c}_i \mid ((\mu(\tilde{c}_i) < \tau) \to \bar{\tilde{e}}) \wedge \\
&\qquad ((\mu(\tilde{c}_i) \geq \tau) \to \tilde{e})] \cup \\
&\quad \mathop{R}_{i=1}^{n} [\bar{\tilde{c}}_i \mid (\mu(\bar{\tilde{c}}_i) \to \bar{\tilde{e}})]\} \vdash \tilde{e} \\
&= \widetilde{C'}(\mathop{R}_{k=1}^{n'} \tilde{c}_k) \vdash \tilde{e},\ \widetilde{C'} \subset \tilde{C}
\end{aligned} \tag{24}$$

where τ is a given threshold of the confidential level, and $\mathop{R}_{i=1}^{n} P_i$ is the *big-R* notation that denotes an iterative behavior or a recurring structure *Pi* [16].

Fuzzy causal inference extends the logical premises of conventional causations to fuzzy expressions constructed with fuzzy set and fuzzy semantics. A fuzzy causal inference can be carried out according to Definition 15 as follows.

Example 13. Assume a fuzzy causal set $\tilde{C}$ that specifies the extents of potential causes for the effect *wet_road* estimated as follows: $\tilde{C}(\tilde{x}, \mu_{\tilde{C}}(\tilde{x})) = \{(rained, 0.95), (sprinkled, 0.6), (flooded, 0.1), (high\_temperature, 0.05)\}$. A validated set of fuzzy causes $\widetilde{C'} \subseteq \tilde{C}$ can be deduced by fuzzy relational causal inference according to Eq. 24:

$$\begin{aligned}
\widetilde{\kappa_{\tilde{C}'}} &= \tilde{C}'(\mathop{R}_{k=1}^{n'} \tilde{c}_k) \vdash \tilde{e} \\
&= \{\mathop{R}_{i=1}^{4} \frac{\partial}{\partial \tilde{c}_i}\{(rained, 0.95), (sprinkled,\ 0.5), (flooded, 0.1), \\
&\qquad (high_temperature, 0)\} \to \tilde{e}\} \vdash wet_road \\
&= \{\mathop{R}_{i=1}^{4} (\tilde{c}_i \mid (\mu_{\tilde{C}}(\tilde{c}_i) \sim \tau = 0.9)) \to \tilde{e})\} \vdash wet_road \\
&= \{\mathop{R}_{i=1}^{4} [\tilde{c}_i \mid ((\mu(\tilde{c}_i) < 0.9) \to \bar{\tilde{e}}) \wedge ((\mu(\tilde{c}_i) \geq 0.9) \to \tilde{e}) \wedge \\
&\qquad \mu(\bar{\tilde{c}}_i) \to \bar{\tilde{e}}]\} \vdash wet_road \\
&= \{rained\} \vdash wet_road
\end{aligned}$$

The fuzzy inference process as illustrated in Example 13 shows that, although there are multiple potential causes for the effect of *wet_road* in various extents, the fuzzy cause *rained* is the only reason when the *fuzzy confidential level* τ is given as 0.9 according to the specific set of fuzzy empirical rules. However, when the threshold is $\tau = 0.5$, the inference result will become $\{rained, sprinkled\} \vdash wet\_road$.

The methodology for fuzzy relational causal inferences can be classified into the categories of *deduction, induction, abduction,* and *analogy* as summarized in Table 1. In Table 1, $\tilde{S}$ is an arbitrary finite nonempty fuzzy set in the universal discourse $\tilde{\mathfrak{U}}_c$, i.e., $\tilde{S} \sqsubseteq \tilde{\mathfrak{U}}_c$, and $p(\tilde{x})$ a proposition where a specific conclusion on $\exists \tilde{a} \in \tilde{S},\ p(\tilde{a})$ is true.

Table 1. Mathematical Models of Fuzzy Inferences

No.	Method	Operator	Fuzzy inference rules
1	Fuzzy deduction	$\tilde{\Downarrow}$	$\forall \tilde{x} \in \tilde{S}(\tilde{x}, \mu_{\tilde{S}}(\tilde{x})), p(\tilde{x}) \vdash q(\tilde{x})$ $\tilde{\Downarrow}\ \exists \tilde{a} \in \tilde{S}(\tilde{a}, \mu_{\tilde{S}}(\tilde{a})), p(\tilde{a}) \vdash q(\tilde{a})$
2.1	Monotonic fuzzy induction	$\hat{\Uparrow}$	$\exists \tilde{a} \in \tilde{S}(\tilde{a}, \mu_{\tilde{S}}(\tilde{a})), p(\tilde{a}) \wedge$ $\exists \tilde{k}, \mathrm{succ}(\tilde{k}) \in \tilde{S}((\tilde{k}, \mu_{\tilde{S}}(\tilde{k})), (\mathrm{succ}(\tilde{k}), \mu(\mathrm{succ}(\tilde{k})))$, $p(\tilde{k}) \vdash p(\mathrm{succ}(\tilde{k}))$ $\hat{\Uparrow}\ \forall \tilde{x} \in \tilde{S}(\tilde{x}, \mu_{\tilde{S}}(\tilde{x})), p(\tilde{x})$
2.2	Random fuzzy induction	$\tilde{\Uparrow}$	$\exists \tilde{a} \in \tilde{S}(\tilde{a}, \mu_{\tilde{S}}(\tilde{a})), p(\tilde{a}) \vdash q(\tilde{a})$ $\wedge \exists \tilde{b} \in \tilde{S}(\tilde{b}, \mu_{\tilde{S}}(\tilde{b})), p(\tilde{b}) \vdash q(\tilde{b})$ $\ldots$ $\wedge \exists \tilde{z} \in \tilde{S}(\tilde{z}, \mu_{\tilde{S}}(\tilde{z})), p(\tilde{z}) \vdash q(\tilde{z})$ $\wedge \exists \tilde{k} \notin \tilde{S}(\tilde{k}, \mu_{\tilde{S}}(\tilde{k})), p(\tilde{k}) \nvdash q(\tilde{k})$ $\tilde{\Uparrow}\ \forall \tilde{x} \in \tilde{S}(\tilde{x}, \mu_{\tilde{S}}(\tilde{x})), p(\tilde{x}) \vdash q(\tilde{x})$
3	Fuzzy abduction	$\tilde{\dashv}$	$\forall \tilde{x} \in \tilde{S}(\tilde{x}, \mu_{\tilde{S}}(\tilde{x})), p(\tilde{x}) \vdash q(\tilde{x}) \wedge r(\tilde{x}) \vdash q(\tilde{x})$ $\tilde{\dashv}\ \exists \tilde{a} \in \tilde{S}(\tilde{a}, \mu_{\tilde{S}}(\tilde{a})), (p(\tilde{a}) \vee r(\tilde{a})) \vdash q(\tilde{a})$
4	Fuzzy Analogy	$\tilde{\propto}$	$\forall \tilde{x} \in \tilde{S}(\tilde{x}, \mu_{\tilde{S}}(\tilde{x})), p(\tilde{x}) \vdash q(\tilde{x})$ $\wedge \exists \tilde{a} \in \tilde{S}(\tilde{a}, \mu_{\tilde{S}}(\tilde{a})), p(\tilde{a}) \vdash q(\tilde{a})$ $\tilde{\propto}\ \exists \tilde{b} \in \tilde{S}(\tilde{b}, \mu_{\tilde{S}}(\tilde{b})) \wedge \tilde{a} \neq \tilde{b}, p(\tilde{b}) \vdash q(\tilde{b})$

The second type of causal inferences is the behavioral fuzzy causal inference, which extends the relational fuzzy causal inference between two fuzzy predicates to the pair of a fuzzy event and a fuzzy behavior.

Definition 16. A *fuzzy behavioral causal inference* $\tilde{\sigma}$ is a dispatch structure that represents a reflexive causation $\widetilde{E'} \hookrightarrow \tilde{B}$ between a set of identified fuzzy events $\widetilde{E'}$ and a set of fuzzy behaviors $\tilde{B}$ by qualitative causal differential on the set of potential fuzzy causal events $\tilde{E} = \mathop{R}\limits_{i=1}^{n}(\tilde{e_i}, \mu_{\tilde{E}}(\tilde{e_i}))$ with respect to $\tilde{e_i}$, $1 \leq i \leq n' = \|\widetilde{E'}\| \leq n = \|\tilde{E}\|$, i.e.:

$$\begin{aligned} \tilde{\sigma} &\triangleq \mathop{R}_{i=1}^{n} [\frac{\partial}{\partial \tilde{e_i}} \tilde{E} \hookrightarrow \tilde{B}] \\ &= \mathop{R}_{k=1}^{\|\widetilde{E'}\|} [\tilde{e_k} \hookrightarrow \tilde{b_k}], \quad \tilde{e_k} \in \tilde{E} \wedge \tilde{b_k} \in \tilde{B} \end{aligned} \tag{25}$$

where the fuzzy events can be classified into the categories of *trigger* events (external) and *perceptive* events (internal) in cognitive computing and fuzzy systems.

All rule-based inferences in the form of *if-then-else* structure are typical instances of fuzzy behavioral causal inferences. Fuzzy finite-state machines and fuzzy automata can also be modeled based on the mathematical model of fuzzy behavioral causal inferences.

Example 14. Human eyes possess three types of visual functions known as *saccade* ($\tilde{b_1}$), *tracking* ($\tilde{b_2}$), and *gaze* ($\tilde{b_3}$) stimulated by corresponding events identified as a *still* object ($\tilde{O_s}$), a *passing* object ($\tilde{O_p}$), and an *approaching* object ($\tilde{O_a}$), respectively. Therefore, the eye mechanisms can be modeled by a pair of fuzzy sets of events $\tilde{E}$ and behaviors $\tilde{B}$ as follows:

$$\widetilde{E} = \{(\widetilde{e_1}, \mu_{\widetilde{E}}(\widetilde{e_1})), (\widetilde{e_2}, \mu_{\widetilde{E}}(\widetilde{e_2})), (\widetilde{e_3}, \mu_{\widetilde{E}}(\widetilde{e_3}))\}$$
$$= \{(\widetilde{O_s}, \mu_{\widetilde{E}}(\widetilde{O_s})), (\widetilde{O_p}, \mu_{\widetilde{E}}(\widetilde{O_p})), (\widetilde{O_a}, \mu_{\widetilde{E}}(\widetilde{O_a}))\}$$

$$\widetilde{B} = \{(\widetilde{b_1}, \mu_{\widetilde{B}}(\widetilde{b_1})), (\widetilde{b_2}, \mu_{\widetilde{B}}(\widetilde{b_2})), (\widetilde{b_3}, \mu_{\widetilde{B}}(\widetilde{b_3}))\}$$

A fuzzy behavioral causal inference for the dispatch functions of eye's movements can be formally described according to Definition 16 as follows:

$$\widetilde{\sigma_{\widetilde{E}'}} = \mathop{R}_{i=1}^{3} [\frac{\partial}{\partial \widetilde{e_i}} \widetilde{E}(\widetilde{e_i}) \hookrightarrow \widetilde{b_i}]$$
$$= \mathop{R}_{i=1}^{3} [(\widetilde{E_t}(\widetilde{e_i}) \oplus \widetilde{E_{t'}}(\widetilde{e_i})) \hookrightarrow \widetilde{b_i}]$$
$$= \begin{cases} (\widetilde{O_s}, \mu_{\widetilde{E}}(\widetilde{O_s} > 0.9) \hookrightarrow \widetilde{B_1}(saccade) \\ (\widetilde{O_p}, \mu_{\widetilde{E}}(\widetilde{O_p} > 0.9) \hookrightarrow \widetilde{B_2}(traking) \\ (\widetilde{O_a}, \mu_{\widetilde{E}}(\widetilde{O_a} > 0.9) \hookrightarrow \widetilde{B_3}(gaze) \end{cases}$$

where the weight of membership for each fuzzy event is determined by the attention capture mechanisms of the brain, and each of the dispatched behaviors can be described as a fuzzy process of eye muscle control and visual information acquisition.

On the basis of the formal causal inference theory, fuzzy semantic inferences can be rigorously manipulated to deal with fuzzy degrees of matters, uncertainties, vague semantics, and fuzzy causality, which enable cognitive machines, cognitive robots, and fuzzy systems to mimic the human intelligent ability and the cognitive processes in cognitive linguistics, fuzzy inferences, cognitive computing, and computational intelligence.

6 Conclusion

This paper has presented a formal theory of fuzzy concepts and fuzzy semantics for formal semantic manipulation and fuzzy causal inferences in cognitive systems and cognitive linguistics. The mathematical models of fuzzy concepts and fuzzy semantics have provided a formal explanation for the fuzzy nature of human language processing and real-time semantics interpretation. It has been identified that the basic unit of linguistic entities that carries unique semantics is a fuzzy concept, which can be modeled as a fuzzy hyperstructure encompassing fuzzy sets of attributes, objects, relations, and qualifications. Complex fuzzy concepts in natural languages have been modeled as a composite fuzzy concept where fuzzy qualifiers are involved to modify the fuzzy semantics of the fuzzy concept by algebraic operations. Two types of fuzzy causal analysis methodologies have been formally modeled known as the relational and behavioral causal inferences between fuzzy cause and effect as well as fuzzy event and behaviors.

This work has demonstrated that fuzzy semantic comprehension is a deductive process, where complex fuzzy semantics can be formally expressed by algebraic operations on elementary ones with fuzzy modifiers. The denotational mathematical structure of fuzzy semantics and fuzzy causal inferences have not only explained the fuzzy nature of linguistic semantics and its comprehension, but also enabled cognitive machines and fuzzy systems to mimic the human fuzzy inference mechanisms in cognitive linguistics, cognitive computing, and computational intelligence.

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