Embedded 5(4) Pair Trigonometrically-Fitted Two Derivative Runge-Kutta Method with FSAL Property for Numerical Solution of Oscillatory Problems

N. SENU, N. A. AHMAD, F. ISMAIL
& N. BACHOK
Institute for Mathematical Research & Department of Mathematics
Universiti Putra Malaysia
43400 Serdang, Selangor
MALAYSIA
norazak@upm.edu.my

Abstract: - Based on First Same As Last (FSAL) technique, an embedded trigonometrically-fitted Two Derivative Runge-Kutta method (TDRK) for the numerical solution of first order Initial Value Problems (IVPs) is developed. Using the trigonometrically-fitting technique, an embedded 5(4) pair explicit fifth-order TDRK method with a "small" principal local truncation error coefficient is derived. The numerical experiments are carried out and showed that our new method is more accurate and efficient when compared with other existing Runge-Kutta (RK) and TDRK methods of the same order.

Key-Words: - Explicit methods, Embedded methods, First Same As Last technique, Initial Value Problems, TDRK methods, Variable step-size

1 Introduction

Consider the numerical solution of the IVPs for the first order Ordinary Differential Equations (ODEs) given in the following form

$$y' = f(x,y), y(x_0) = y_0. \tag{1}$$

A numerous number of researchers have proposed several highly efficient TDRK methods with constant and variable step-sizes as well as implementing the First Same As Last (FSAL) technique in the derivation of their methods. Chan and Tsai[1] introduced special explicit TDRK methods by including the second derivative which involves one evaluation of f and a few evaluations of g per step. They presented methods with stages up to five and order up to seven. Chan et al.[2] then presented their study related to stiff ODEs problems on explicit and implicit TDRK methods and extend the applications of the TDRK methods to various Partial Differential Equations (PDEs).

Besides, Zhang et al.[3] derived a new fifth-order trigonometrically-fitted TDRK method for the numerical solution of the radial Schrödinger equation and oscillatory problems. Meanwhile, two fourth-order and three practical exponentially-fitted TDRK methods were presented by Fang et al.[4] and Chen et al.[5] respectively. The new methods were compared with some famous optimized codes and traditional exponentially-fitted Runge-Kutta (RK) methods in the literature.

Bogacki and Shampine[6] derived a 3(2) pair of RK formula while Dormand and Prince[7] introduced a family of embedded RK5(4) which have extended stability regions and a 'small' fifth-order principal truncation terms in a few decades back. Tsitouras[8] presented RK pairs of order 5(4) which satisfy only the first column simplifying assumption by neglecting the row simplifying assumptions. Fang et al.[9] designed new embedded pairs of higher order RK methods specially adapted to the numerical integration of IVPs with oscillatory solutions by implementing the FSAL property.

Chen et al.[10] improved traditional RK methods by introducing frequency-depending weights in the update. With the phase-fitting and amplification-fitting conditions and algebraic order conditions, new practical RK integrators are obtained and two of the new methods have updates that are also phase-fitted and amplification-fitted. Recently, in [11] and [12], Demba et al. derived four-stage fourth-order explicit trigonometrically-fitted Runge-Kutta-Nyström (RKN) method and fifth-order four-stage explicit trigonometrically-fitted RKN method respectively for the numerical solution of second-order initial value problems with oscillatory solutions based on Simos' RKN method.

Meanwhile, Fawzi et al. in their papers [13] and [14] developed fourth-order phase-fitted and amplification-fitted modified RK method and fourth-order seven stage phase-fitted and amplification-fitted RK methods respectively.

Motivated by the FSAL property as well as the trigonometrically-fitting technique, hence in this paper, a new fifth-order four stages 5(4) pair trigonometrically-fitted TDRK method with FSAL property is constructed. An overview of TDRK method will be given in the Section 2. In Section 3, trigonometrically-fitted properties are considered. The derivation of the new method is discussed in Section 4. Finally, the numerical results, discussion and conclusion are dealt in Section 5, Section 6 and Section 7 respectively.

2 Two Derivative Runge-Kutta Method

Consider the IVPs (1) with $f: \Re^N \to \Re^N$. In this case, we also assume that the second derivative is known where

$$y'' = g(y) := f'(y)f(y), g: \Re^N \to \Re^N. \tag{2}$$

An explicit TDRK method for the numerical integration of IVPs is given by

$$Y_i = y_n + h\sum_{j=1}^{s} a_{ij}\, f(y_j) + h^2 \sum_{j=1}^{s} \hat{a}_{ij}\, g(Y_j), \tag{3}$$

$$y_{n+1} = y_n + h\sum_{i=1}^{s} b_i\, f(y_i) + h^2 \sum_{i=1}^{s} \hat{b}_i\, g(Y_i), \tag{4}$$

where $i = 1, \dots, s$.

Consider the following explicit methods which have a minimal number of function evaluations where

$$Y_i = y_n + hc_i f(x_n, y_n) + h^2 \sum_{j=1}^{i-1} \hat{a}_{ij}\, g(x_n + hc_j, Y_j), \tag{5}$$

$$y_{n+1} = y_n + hf(x_n, y_n) + h^2 \sum_{i=1}^{s} \hat{b}_i\, g(x_n + hc_i, Y_i), \tag{6}$$

where $i = 2, \dots, s$.

This method with a minimal number of function evaluations is called special explicit TDRK method. The difference between this method with the traditional RK methods is that special explicit TDRK methods involves only one evaluation of f per step. Butcher tableau below shows the difference between the explicit TDRK method and special explicit TDRK method

Table 1 Difference between Butcher tableau for explicit and special explicit TDRK method.

$$\begin{array}{c|c|c} c & A & \hat{A} \\ \hline & b^T & \hat{b}^T \end{array} \rightarrow \begin{array}{c|c} c & \hat{A} \\ \hline & \hat{b}^T \end{array}$$

The TDRK parameters $\hat{a}_{ij}, \hat{b}_i$ and c_j are assumed to be real and s is the number of stages of the method. We introduce the s-dimensional vectors $\hat{b}, \bar{b}, c$ and $s \times s$ matrix, $\hat{A}$ where $\hat{b} = [\hat{b}_1, \hat{b}_2, \dots, \hat{b}_s]^T, \bar{b} = [\bar{b}_1, \bar{b}_2, \dots, \bar{b}_s]^T, c = [c_1, c_2, \dots, c_s]^T$ and $\hat{A} = [\hat{a}_{ij}]$ respectively. An embedded $r(m)$ pair of TDRK method is basedon the method $(c, \hat{A}, \hat{b})$ of order r and the other TDRK method $(c, \hat{A}, \bar{b})$ of order m where $m < r$ and it can be expressed in the following Butcher tableau

Table 2 Butcher tableau for embedded TDRK method.

$$\begin{array}{c|c} c & \hat{A} \\ \hline & \hat{b}^T \\ \hline & \bar{b}^T \end{array}$$

In embedded pair of explicit TDRK methods, our main target is to achieve an affordable and cheap local error estimation which will be used in a variable step-size algorithm. At the point x_{n+1} the local error estimation is determined by the expression

$$\delta_{n+1} = \bar{y}_{n+1} - y_{n+1} \tag{7}$$

where $\bar{y}_{n+1}$ is the solution using the fourth-order formula. The step-size h are being controlled using the procedure given as follows:

- If $TOL/div < Est < div.TOL$ then $h_{n+1} = h_n$
- If $Est \le TOL/div$ then $h_{n+1} = 2h_n$
- If $Est \ge TOL.div$ then $h_{n+1} = \frac{1}{2}h_n$ and repeat the step

where $div = 2^{17}, TOL$ is the accuracy required which is the maximum allowable local error and $Est = \|\delta_{n+1}\|_2$ represents the local error estimation at each step. This procedure do not allow step-size change after each step as it would lead to unnecessary rounding errors. If the step is acceptable $(TOL/div < Est < div.TOL$ and $Est \leq TOL/div)$ then we applied the accepted procedure of performing local extrapolation (or higher order mode).

3 Trigonometrically-Fitted Two Derivative Runge-Kutta Method Technique

When the special explicit TDRK method (5) and (6) is applied to the ODEs (1) with $y'' = -\lambda^2 y$, the method becomes

$$y_{n+1} = y_n + hf(x_n, y_n) + h^2 \sum_{i=1}^{s} \hat{b}_i\, g(x_n + c_i h, Y_i), \quad (8)$$

where

$$Y_1 = y_n + c_1 hf(x_n, y_n), \quad (9)$$
$$Y_2 = y_n + c_2 hf(x_n, y_n) - v^2 \hat{a}_{21} Y_1, \quad (10)$$
$$Y_3 = y_n + c_3 hf(x_n, y_n) - v^2(\hat{a}_{31} Y_1 + \hat{a}_{32} Y_2), \quad (11)$$
$$Y_4 = y_n + c_4 hf(x_n, y_n) - v^2(\hat{a}_{41} Y_1 + \hat{a}_{42} Y_2 + \hat{a}_{43} Y_3), \quad (12)$$
$$\vdots$$
$$Y_i = y_n + c_i hf(x_n, y_n) + h^2 \sum_{j=1}^{s} \hat{a}_{ij}\, (-\lambda^2 Y_i). \quad (13)$$

which results in

$$y_{n+1} = y_n + hf(x_n, y_n) + h^2 \sum_{i=1}^{s} \hat{b}_i\, (-\lambda^2 Y_j). \quad (14)$$

Let $y_n = e^{i\lambda x}$ and $f(x_n, y_n) = i\lambda y_n$, compute the values for y_{n+1} and substitute those values in equation (8)-(14). By using $e^{iv} = \cos(v) + i\sin(v)$, the following equation is obtained

$$\cos(v) + i\sin(v) = 1 + z + z^2 \sum_{i=1}^{s} \hat{b}_i \left(1 + zc_i + z^2 \sum_{j=1}^{i-1} \hat{a}_{ij}\, Y_j e^{-I\lambda x_n} \right), \quad (15)$$

4 Construction of The New Method

In this section, the embedded fourth-order method will be derived using the existing fifth-order TDRK method given by Chan and Tsai[1]. The TDRK parameters must satisfy the following order conditions.

Table 3 Order conditions for TDRK methods.

Order	Conditions	
2	$\hat{b}^T e = \frac{1}{2}$	
3	$\hat{b}^T c = \frac{1}{6}$	
4	$\hat{b}^T c^2 = \frac{1}{12}$	
5	$\hat{b}^T c^3 = \frac{1}{20}$	$\hat{b}^T \hat{A} c = \frac{1}{120}$
6	$\hat{b}^T c^4 = \frac{1}{30}$	$\hat{b}^T c \hat{A} c = \frac{1}{180}$
	$\hat{b}^T \hat{A} c^2 = \frac{1}{360}$	

The following simplifying assumption is also used in practice.

$$\sum_{i=1}^{s} \hat{a}_{ij} = \frac{1}{2} {c_i}^2 \text{ for } i = 2, \ldots, s. \quad (16)$$

The FSAL property is given as follows

$$\hat{b}_i = \hat{a}_{si} \text{ where } i = 1, \ldots, s-1 \text{and } \hat{b}_s = 0. \quad (17)$$

The advantage of FSAL technique is the last function evaluation of a step is used as the first function evaluation of the next step.

4.1 The Fifth-Order Formula

For the fifth-order TDRK method, the existing fifth-order method given by Chan and Tsai[1] is used. The derivation of this method is very simple. To get a TDRK formula of algebraic order five, there will be five equations and six unknowns. Hence,this system has one free parameter. Solve the order conditions given by Table 3 and this will lead to the following equation in term of c_2

$$\hat{a}_{32} = \frac{1}{250} \frac{(5\,c_2 - 3)(-10\,c_2 + 3 + 10\,{c_2}^2)}{(2\,c_2 - 1)^3 c_2}, \quad (18)$$
$$\hat{b}_1 = \frac{1}{12} \frac{10\,{c_2}^2 - 8\,c_2 + 1}{c_2(5\,c_2 - 3)}, \quad (19)$$

$$\hat{b}_2 = -\frac{1}{6}\frac{-c_2+1/2}{c_2(2\,c_2-1)(-10\,c_2+3+10\,{c_2}^2)}, \quad (20)$$
$$\hat{b}_3 = \frac{25}{12}\frac{(2\,c_2-1)^3}{(5\,c_2-3)(-10\,c_2+3+10\,{c_2}^2)}, \quad (21)$$
$$c_3 = \frac{1}{5}\frac{5\,c_2-3}{2\,c_2-1}. \quad (22)$$

Let $c_2 = \frac{1}{10}$, hence coefficients of this three stage fifth-order formula are given by the following Table 4.

Table 4 Three stage fifth-order TDRK method.

0	0		
$\frac{3}{10}$	$\frac{9}{200}$	0	
$\frac{3}{4}$	0	$\frac{9}{32}$	0
	$\frac{5}{24}$	$\frac{25}{81}$	$\frac{8}{81}$

The norms of the principal local truncation error coefficients for y_n is given by

$$\left\|\tau^{(6)}\right\|_2 = 8.5616861162 \times 10^{-4}. \quad (23)$$

4.2 The Fourth-Order Formula

Based on the values of $\hat{A}$ and c given in the previous section, a four stages fourth-order embedded formula will be derived. Implementing the FSAL property, $c_4 = 1, \hat{a}_{41} = \hat{b}_1, \hat{a}_{42} = \hat{b}_2, \hat{a}_{43} = \hat{b}_3$ and $\hat{a}_{44} = 0$.

According to order conditions in Table 3 for second, third-order and fourth-order, we have

$$\bar{b}_1 + \bar{b}_2 + \bar{b}_3 + \bar{b}_4 - \frac{1}{2} = 0, \quad (24)$$
$$\frac{3}{10}\,\bar{b}_2 + \frac{3}{4}\,\bar{b}_3 + \bar{b}_4 - \frac{1}{6} = 0, \quad (25)$$
$$\frac{9}{100}\,\bar{b}_2 + \frac{9}{16}\,\bar{b}_3 + \bar{b}_4 - \frac{1}{12} = 0. \quad (26)$$

Solving equation (24)-(26) will lead to a solution of $\bar{b}_1, \bar{b}_2$ and $\bar{b}_3$ in term of $\bar{b}_4$

$$\bar{b}_1 = \frac{5}{54} - \frac{7}{9}\,\bar{b}_4, \quad (27)$$
$$\bar{b}_2 = \frac{25}{81} + \frac{50}{27}\,\bar{b}_4, \quad (28)$$
$$\bar{b}_3 = \frac{8}{81} - \frac{56}{27}\,\bar{b}_4. \quad (29)$$

Our main objective is to choose $\bar{b}_4$ such that the principal local truncation error coefficient, $\left\|\tau^{(5)}\right\|_2$ have very small value. Wrong choices of $\bar{b}_4$ may cause a huge difference global error compared with the error tolerance specified. By plotting the graph of $\left\|\tau^{(5)}\right\|_2$ against $\bar{b}_4$ and choosing a small value of $\bar{b}_4$ in the range of [0.0,1.0], hence the value of $\bar{b}_4$ lies between [0.0,0.1]. $\bar{b}_4 = 11/2000$ is chosen for an optimized pair. All the coefficients are showed in Table 5 and it is denoted as FLTDRK5(4).

Table 5 Butcher Tableau for FLTDRK5(4) method.

0	0			
$\frac{3}{10}$	$\frac{9}{200}$	0		
$\frac{3}{4}$	0	$\frac{9}{32}$	0	
1	$\frac{5}{24}$	$\frac{25}{81}$	$\frac{8}{81}$	0
	$\frac{5}{24}$	$\frac{25}{81}$	$\frac{8}{81}$	0
	$\frac{4769}{54000}$	$\frac{1033}{3240}$	$\frac{1769}{20500}$	$\frac{11}{2000}$

The norms of the principal local truncation error coefficients for $\bar{y}_n$ is given by

$$\left\|\tau^{(5)}\right\|_2 = 9.635906518 \times 10^{-4}. \quad (30)$$

4.3 Derivation of Embedded 5(4) Pair Trigonometrically-Fitted TDRK Method

In deriving the new method, the trigonometrically-fitted property will be applied to the embedded TDRK method derived earlier where the coefficient is given in Table 5. Hence, the derivation of the new embedded trigonometrically-fitted TDRK method will be discussed next.

For optimized value of maximum global error, the combination of $\hat{b}_2$ and $\hat{b}_4$ are chosen as free parameters. By using the coefficients of the higher order method (fifth-order), separate the real part and the imaginary part of (15) and this leads to

$$\cos(v) = 1 + \frac{1}{800}\hat{b}_4 v^8 + \left(-\frac{1}{800} - \frac{1}{24}\hat{b}_4\right)v^6 + \left(\frac{9}{200}\hat{b}_2 + \frac{1}{36} + \frac{1}{2}\hat{b}_4\right)v^4 + \left(-\frac{31}{162} - \hat{b}_2 - \hat{b}_4\right)v^2, \quad (31)$$

$$\sin(v) = -\frac{1}{120}\hat{b}_4 v^7 + \left(\frac{1}{120} + \frac{1}{6}\hat{b}_4\right)v^5 + \left(-\frac{3}{10}\hat{b}_2 - \frac{2}{27} - \hat{b}_4\right)v^3 + v. \quad (32)$$

Solving equation (31) and (32) we will obtain

$$\hat{b}_2 = -\frac{1}{324v^3(2v^4 + 420 - 37v^2)}(1620v^5\cos(+ 3700v^5 - 200v^7 - 8100v^4\sin(v) + 243v^6\sin(v) + 194400\cos(v)v - 42000v^3 - 194400\sin(v) + 97200v^2\sin(v) - 32400\cos(v)v^3), \quad (33)$$

$$\hat{b}_4 = \frac{1}{v^3(2v^4 + 420 - 37v^2)}(180\cos(v)v + 420v - 37v^3 + 2v^5 - 600\sin(v) + 27v^2\sin(v)). \quad (34)$$

As $v \to 0$, the following Taylor expansions are obtained

$$\hat{b}_2 = \frac{25}{81} - \frac{1}{11760}v^4 + \frac{3097}{44452800}v^6 + \frac{139709}{205371936000}v^8 - \frac{111256711}{1121330770560000}v^{10} - \frac{6929656837}{470958923635200000}v^{12} + \cdots,$$

$$\hat{b}_4 = \frac{79}{352800}v^4 + \frac{3037}{222264000}v^6 + \frac{477013}{2053719360000}v^8 - \frac{510764327}{11213307705600000}v^{10} - \frac{24076649309}{4709589236352000000}v^{12} + \cdots.$$

As $v \to 0$, the coefficients of $\hat{b}_2$ and $\hat{b}_4$ of the higher order method will reduce to its original coefficient that is the higher order method.

In a similar way, for optimized value of maximum global error, the combination of $\bar{b}_3$ and $\bar{b}_4$ are chosen as free parameters. By using the coefficients of the lower order method (fourth-order), separate the real part and the imaginary part of (15) and this leads to

$$cos(v) = 1 + \frac{1}{800}\bar{b}_4 v^8 + \left(-\frac{81}{6400}\bar{b}_3 - \frac{1}{24}\bar{b}_4\right)v^6 + \left(\frac{9}{32}\bar{b}_3 + \frac{1033}{72000} + \frac{1}{2}\bar{b}_4\right)v^4 + \left(-\frac{65957}{162000} - \bar{b}_4 - \bar{b}_3\right)v^2, \quad (35)$$

$$sin(v) = -\frac{1}{120}\bar{b}_4 v^7 + \left(\frac{1}{6}\bar{b}_4 + \frac{27}{320}\bar{b}_3\right)v^5 + \left(-\frac{3}{4}\bar{b}_3 - \frac{1033}{10800} - \bar{b}_4\right)v^3 + v. \quad (36)$$

Solving equation (35) and (36) we will obtain

$$\bar{b}_3 = \frac{2}{2025v^3(-4800 - 220v^2 + 107v^4)}(10667 + 24300v^6\sin(v) - 20350v^5 + 162000v^5\cos(v) - 810000v^4\sin(v) - 424560v^3 - 3240000\cos(v)v^3 + 9720000v^2\sin(v) + 19440000\cos(v)v - 19440000\sin(v)), \quad (37)$$

$$\bar{b}_4 = \frac{1}{100v^3(-4800 - 220v^2 + 107v^4)}(-1440000\,cos(v)v + 162000\cos(v)v^3 - 480000v - 24640v^3 + 10667v^5 + 1920000\sin(v) - 540000v^2\sin(v) + 24300v^4\sin(v)). \quad (38)$$

As $v \to 0$, the following Taylor expansions are obtained

$$\bar{b}_3 = \frac{1769}{20250} + \frac{11}{60000}v^2 - \frac{9959}{20160000}v^4 + \frac{17619839}{72576000000}v^6 - \frac{7386874259}{191600640000000}v^8 + \frac{83281063591}{10868981760000000}v^{10} - \frac{174784988603783}{14347055923200000000}v^{12} + \cdots,$$

$$\bar{b}_4 = \frac{11}{2000} - \frac{11}{60000}v^2 + \frac{11379}{22400000}v^4 - \frac{20368771}{145152000000}v^6 + \frac{8355100183}{383201280000000}v^8 - \frac{5017304507021}{119558799360000000}v^{10} + \frac{1364007862423069}{20085878292480000000}v^{12} + \cdots.$$

As $v \to 0$, the coefficients of $\bar{b}_3$ and $\bar{b}_4$ of the lower order method will reduce to its original coefficient that is the higher order method. The above two solutions are the new embedded 5(4) trigonometrically-fitted TDRK method denoted as FSALTDRK5(4).

5 Problems Tested and Numerical Results

The performance of the studied method FSALTDRK5(4) are compared with existing embedded RK and TDRK methods by considering the following problems. All problems below are tested using C code for solving differential equations. The tolerances used are $TOL = 10^{-2i}, i = 1, \ldots, 5$.

Problem 1 (Senu[15])

$$y_1' = y_2, \quad y_1(0) = 1.1,$$
$$y_2' = -16y_1 + 116e^{-10x}, \quad y_2(0) = -10,$$
$$y_3' = y_4, \quad y_3(0) = 1,$$
$$y_4' = -16y_3 + 116e^{-10x} \quad y_4(0) = -9.6,$$
$$x \in [0,100].$$

Exact solution is

$$y_1(t) = 0.1\cos(4x) + e^{-10x},$$
$$y_2(x) = -0.4\ \sin(4x) - 10e^{-10x},$$
$$y_3(t) = 0.1\sin(4x) + e^{-10x},$$
$$y_4(x) = 0.4\cos(4x) - 10e^{-10x}.$$

Problem 2 (Rabiei and Ismail[16])

$$y_1' = -2y_1 + y_2 + 2\sin(x),$$
$$y_2' = y_1 - 2y_2 + 2(\cos(x) - \sin(x)),$$
$$y_1(0) = 2, \quad y_2(0) = 3, \quad x \in [0,100].$$

Exact solution is

$$y_1(x) = 2e^{-x} + \sin(x),$$
$$y_2(x) = 2e^{-x} + \cos(x).$$

Problem 3 (Duffing problem[17])

$$y_1' = y_2,$$
$$y_2' = -y_1 - y_1^3 + 0.002\cos(1.01x),$$
$$y_1(0) = 0.200426728067, \quad x \in [0,100].$$
$$y_2(0) = 0,$$

Exact solution is

$$y_1(x) = 0.200179477536\cos(1.01x) + 2.46946143 \times 10^{-4}\cos(3.03x) + 3.04014 \times 10^{-7}\cos(5.05x) + 3.74 \times 10^{-10}\cos(7.07x),$$
$$y_2(x) = -0.2021812723\sin(1.01x) - 7.482468133 \times 10^{-4}\sin(3.03x) - 1.53527070 \times 10^{-6}\sin(5.05x) - 2.64418 \times 10^{-9}\sin(7.07x).$$

Problem 4 (Prothero-Robinson problem[1])

$$y' = \lambda(y - \varphi) + \varphi',$$
$$y(0) = \varphi(0), \quad Re(\lambda) < 0, \quad x \in [0,100].$$

where $\varphi(x)$ is a smooth function. We take $\lambda = -1$ and $\varphi(x) = sin(x)$.

Exact solution is $y(x) = \varphi(x)$.

The following notations are used in Figures 1–4 :

- **FSALTDRK5(4):** New embedded 5(4) pair trigonometrically-fitted TDRK method with FSAL property of four stages derived in this paper.
- **TDRK4(5):** Existing embedded 5(4) pair TDRK method five stages derived by Chan and Tsai[1].
- **RKDP5(4)**: Existing embedded 5(4) pair RK method seven stages given in Dormand and Prince[18].

- **RKF4(5)**: Existing embedded 5(4) pair RK method six stages given in Bu *et al.*[19].
- **RKCK5(4)**: Existing embedded 5(4) pair RK method six stages given in Press *et al.*[20].

The performance of these numerical results are represented graphically in the following Figures 1–4:

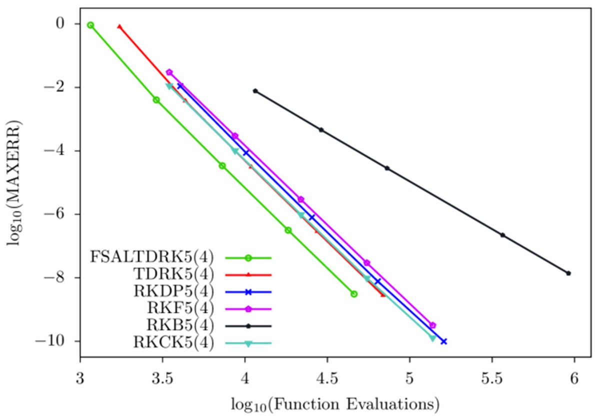

Figure 1 Efficiency graph for Problem 1

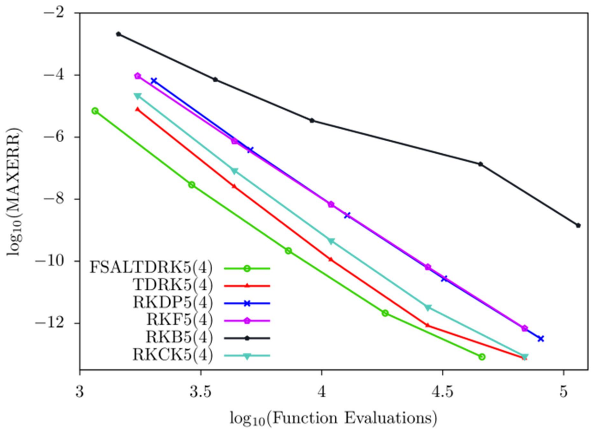

Figure 2 Efficiency graph for Problem 2

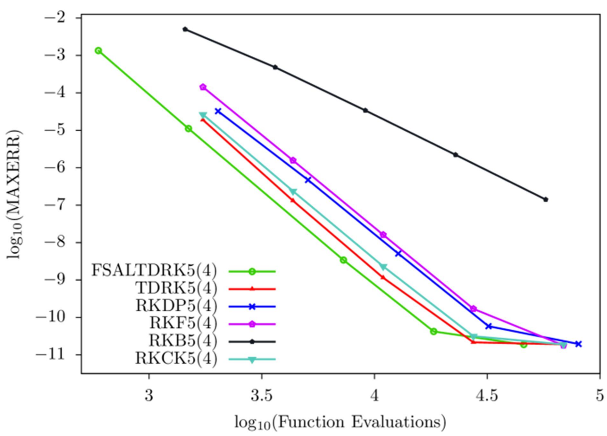

Figure 3 Efficiency graph for Problem 3

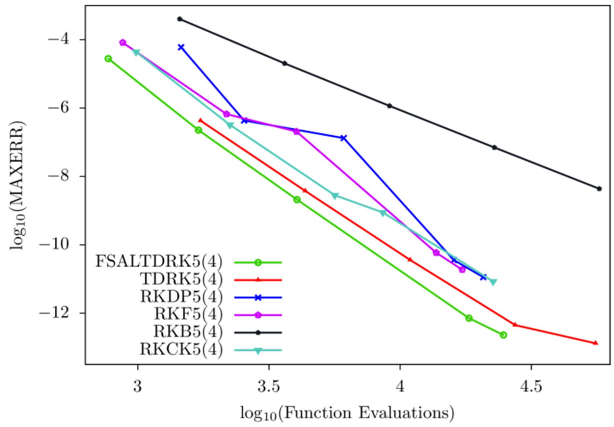

Figure 4 Efficiency graph for Problem 4

6 Discussion

The results show the typical properties of the new embedded trigonometrically-fitted TDRK method with FSAL property, FSALTDRK5(4) which have been derived earlier. The derived method is compared with some well-known existing embedded RK and TDRK methods. Figures 1-4 represent the efficiency and accuracy of the method developed by plotting the graph of the logarithm of the maximum global error against the logarithm number of function evaluations. From Figures 1–2, the global error produced by the FSALTDRK5(4) method has smaller global error compared to TDRK5(4), RKDP5(4), RKF5(4) and RKCK5(4).

Meanwhile, in Figures 3-4, FSALTDRK5(4) has a bigger global error compared to TDRK5(4) but FSALTDRK5(4) has fewer number of function evaluations per step compared to TDRK5(4). All in all, FSALTDRK5(4) has the least number of function evaluations compared to other existing embedded RK and TDRK methods of the same order.

7 Conclusion

In this research, an embedded 5(4) pair trigonometrically-fitted TDRK method with FSAL property has been developed. Based on the numerical results obtained, it can be concluded that the new 5(4) pair FSALTDRK5(4) method is more promising compared to other well-known existing explicit embedded RK and TDRK methods in terms of accuracy and the number of function evaluations.

References:

[1] Chan, R. P., & Tsai, A. Y. 2010. On explicit two-derivative Runge-Kutta methods. Numerical Algorithms, 53(2-3), 171-194.

[2] Chan, R. P., Wang, S., & Tsai, A. Y. 2012, September. Two-derivative Runge-Kutta methods for differential equations. In Numerical Analysis and Applied Mathematics ICNAAM 2012: International Conference of Numerical Analysis and Applied Mathematics (Vol. 1479, No. 1, pp. 262-266). AIP Publishing.

[3] Chan, R. P., Wang, S., & Tsai, A. Y. 2012, Zhang, Y., Che, H., Fang, Y., & You, X. 2013. A new trigonometrically fitted two-derivative Runge-Kutta method for the numerical solution of the Schrödinger equation and related problems. Journal of Applied Mathematics, 2013.

[4] Fang, Y., You, X., & Ming, Q. 2013. Exponentially fitted two-derivative runge–kutta methods for the schrödinger equation. International Journal of Modern Physics C, 24(10), 1350073.

[5] Chen, Z., Li, J., Zhang, R., & You, X. 2015. Exponentially Fitted Two-Derivative Runge-Kutta Methods for Simulation of Oscillatory Genetic Regulatory Systems. Computational and mathematical methods in medicine, 2015.

[6] Bogacki, P., & Shampine, L. F. 1989. A 3 (2) pair of Runge-Kutta formulas. Applied Mathematics Letters, 2(4), 321-325.

[7] Dormand, J. R., & Prince, P. J. 1980. A family of embedded Runge-Kutta formulae. Journal of computational and applied mathematics, 6(1), 19-26.

[8] Tsitouras, C. 2011. Runge–Kutta pairs of order 5 (4) satisfying only the first column simplifying assumption. Computers & Mathematics with Applications, 62(2), 770-775.

[9] Fang, Y., Song, Y., & Wu, X. 2008. New embedded pairs of explicit Runge–Kutta methods with FSAL properties adapted to the numerical integration of oscillatory problems. Physics Letters A, 372(44), 6551-6559.

[10] Chen, Z., You, X., Shu, X., & Zhang, M. 2012. A new family of phase-fitted and amplification-fitted Runge-Kutta type methods for oscillators. Journal of Applied Mathematics, 2012.

[11] Demba, M. A., Senu, N., & Ismail, F. 2016. Trigonometrically-fitted explicit four-stage fourth-order Runge–Kutta–Nyström method for the solution of initial value problems with oscillatory behavior. Global Journal of Pure and Applied Mathematics, 12(1), 67-80.

[12] Demba, M. A., Senu, N., & Ismail, F. 2016. Fifth-Order Four-Stage Explicit Trigonometrically-Fitted Runge–Kutta–Nyström Methods. In Recent Advances in Mathematical Sciences (pp. 27-36). Springer Singapore.

[13] Fawzi, F. A., Senu, N., Ismail, F., & Majid, Z. A. (2015, August). A Phase-Fitted and Amplification-Fitted Modified Runge-Kutta Method of Fourth Order for Periodic Initial Value Problems. In Research and Education in Mathematics (ICREM7), 2015 International Conference on (pp. 25-28). IEEE.

[14] F. A. Fawzi, N. Senu, F. Ismail, and Z. A. Majid. 2016. A New Efficient Phase-Fitted and Amplification-Fitted Runge-Kutta Method for Oscillatory Problems. International Journal of Pure and Applied Mathematics, 107, 69-86.

[15] Senu, N. 2010. Runge-Kutta-Nystrom Methods for Solving Oscillatory Problems (Doctoral dissertation, Universiti Putra Malaysia).

[16] Rabiei, F., & Ismail, F. 2012. Fifth-order Improved Runge-Kutta method for solving ordinary differential equation. Australian Journal of Basic and Applied Sciences, 6(3), 97-105.

[17] Kosti, A. A., Anastassi, Z. A., & Simos, T. E. 2012. An optimized explicit Runge–Kutta–Nyström method for the numerical solution of orbital and related periodical initial value problems. Computer Physics Communications, 183(3), 470-479.

[18] Dormand, J. R., & Prince, P. 1978. New Runge-Kutta algorithms for numerical simulation in dynamical astronomy. Celestial Mechanics, 18(3), 223-232.

[19] Bu, S., Chung, W., & Kim, P. 2014. A Method for Improving the Embedded Runge-Kutta Fehlberg 4 (5).

[20] Press, W. H., Teukolsky, S. A., Vetterling, W. T., & Flannery, B. P. 1996. Numerical recipes in C (Vol. 2). Cambridge: Cambridge university press.