

New Space/Wavelength/Time optical codes for OCDMA

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Abstract: New constructions of 3-dimensional optical orthogonal codes are presented. In each case the codes have ideal autocorrelation $\lambda_a = 0$, and cross correlation of $\lambda_c = 1$. All codes produced are demonstrated to be optimal. The constructions utilize a particular automorphism (a Singer cycle) of $\text{PG}(k, q)$, the finite projective geometry of dimension k over the field of order q , or its affine analogue in $AG(k, q)$.

Key-Words: 3-D code, 3-D OOC, Optical Orthogonal Codes, Johnson bound, finite projective geometries, $\text{PG}(k, q)$, Singer cycle. optimal codes

1 Introduction

Optical code division multiple access (OCDMA) continues to be of great interest among multiple access systems due to ease of implementation, support for asynchronous and secure communication, soft traffic handling capability, and strong performance with high numbers of users [1]. The work of Salehi *et. al.* [2] [3], spearheaded the use of optical orthogonal codes for OCDMA, and these codes continue to be highly effective over a quarter of a century later.

An $(n, w, \lambda_a, \lambda_c)$ -optical orthogonal code (OOC) is a family of (1-dimensional) binary sequences (codewords) of length n , and constant Hamming weight w satisfying the following two conditions:

- (auto-correlation property) for any codeword $c = (c_0, c_1, \dots, c_{n-1})$ and for any integer $1 \leq t \leq n - 1$, we have $\sum_{i=1}^{n-1} c_i c_{i+t} \leq \lambda_a$,
- (cross-correlation property) for any two distinct codewords c, c' and for any integer $0 \leq t \leq n - 1$, we have $\sum_{i=0}^{n-1} c_i c'_{i+t} \leq \lambda_c$,

where each subscript is reduced modulo n .

An $(n, w, \lambda_a, \lambda_c)$ -OOC C with $\lambda_a = \lambda_c$ is denoted an (n, w, λ) -OOC. The number of codewords is the *size* or *capacity* of the code, denoted $|C|$. For fixed values of n, w, λ_a and λ_c , the largest size of an $(n, w, \lambda_a, \lambda_c)$ -OOC is denoted $\Phi(n, w, \lambda_a, \lambda_c)$. An $(n, w, \lambda_a, \lambda_c)$ -OOC is said to be *optimal* if $|C| = \Phi(n, w, \lambda_a, \lambda_c)$. Optimal OOCs facilitate the largest

possible number of asynchronous users to transmit information efficiently and reliably.

A limitation of 1-D OOCs is that the autocorrelation cannot be zero, and to maintain minimal autocorrelation of 1 the code length must increase rapidly with the number of users. The 1-D-OOCs spread input data bits only in the time domain. Technologies such as wavelength-division-multiplexing (WDM) and dense-WDM enable the spreading of codewords in both space and time [4], or in wavelength and time [5]. Hence, codewords may be considered as $\Lambda \times T$ (0, 1)-matrices. These codes are referred to in the literature as multiwavelength, multiple-wavelength, wavelength-time hopping, and 2-dimensional OOCs (2D-OOCs). An additional dimension permits codes with at most a single pulse per row, yielding autocorrelation zero and thereby improving the OCDMA performance in comparison with 1-D OCDMA. For optimal constructions of 2-D OOC's see [6, 7, 8]. Later, a third dimension was added which gave an increase the code size and the performance of the code [9]. In 3-D OCDMA the optical pulses are spread in three domains space, wave-length, and time, with codes referred to as *space/wavelength/time spreading* codes, or *3-D OOC*.

1.1 3-D OOCs and Bounds

We denote by $(\Lambda \times S \times T, w, \lambda_a, \lambda_c)$ a 3D-OOC with constant weight w , Λ wavelengths, space spreading length S , and time-spreading length T (hence, each codeword may be considered as an $\Lambda \times S \times T$ binary array). The autocorrelation and cross correlation of an $(\Lambda \times S \times T, w, \lambda_a, \lambda_c)$ -3D-OOC have the following properties.

- (auto-correlation property) for any codeword $A = (a_{i,j,k})$ and for any integer $1 \leq t \leq T - 1$, we have $\sum_{i=0}^{S-1} \sum_{j=0}^{\Lambda-1} \sum_{k=1}^{T-1} a_{i,j,k} a_{i,j,k+t} \leq \lambda_a$,
- (cross-correlation property) for any two distinct codewords $A = (a_{i,j,k})$, $B = (b_{i,j,k})$ and for any integer $0 \leq t \leq T - 1$, we have $\sum_{i=0}^{S-1} \sum_{j=0}^{\Lambda-1} \sum_{k=0}^{T-1} a_{i,j,k} b_{i,j,k+t} \leq \lambda_c$,

where each subscript is reduced modulo T . There are practical considerations to be made with regard to the implementation of these codes. First, in optical code-division multiple-access (OCDMA) applications, minimal correlation values are most desirable. Implementation is simplified (and more cost effective) when $\lambda_a = 0$ [10]. Codes satisfying $\lambda_a = 0$ will be said to be *ideal* here. Ideal codes with minimal autocorrelation $\lambda_c = 1$ are our main focus.

A wavelength/time plane is called a *spatial plane*, a space/time plane is called a *wavelength plane*, and a space/wavelength plane is called a *temporal plane*. One way to achieve $\lambda_a = 0$ is to select codes with at most one pulse per spatial plane. Such codes are referred to as *at most one pulse per plane* (AMOPP) codes. AMOPP codes of maximal weight S have a single pulse per spatial plane, and are referred to as SPP codes. Codes with at most one pulse per wavelength plane also enjoy zero autocorrelation, and are denoted AMOPW codes. AMOPW codes of maximal weight Λ are single pulse per wavelength (SPW) codes. Codes with at most one (resp. exactly one) pulse per temporal plane do not necessarily have $\lambda_a = 0$ are referred to as AMOPT and SPT codes respectively. As it is of interest to construct codes with as large cardinality as possible, we now discuss some upper bounds on the size of codes.

In order to develop new bounds for codes with ideal autocorrelation we introduce the notion of Hamming correlation. Given two 1-dimensional codewords over any alphabet, the *Hamming correlation* is the number of non-zero agreements between the two codewords. By an $(n, w, \lambda)_{m+1}$ -code, we denote a code of length n , with constant weight w , and maximum Hamming correlation λ over an alphabet of size $m+1$ (containing zero). For binary codes ($m = 1$) the subscript 2 is typically dropped. Let $A(n, w, \lambda)_{m+1}$ denote the maximum size of an $(n, w, \lambda)_{m+1}$ -code. The bound of Johnson [11] establishes the following bound in the binary case.

Theorem 1 (Johnson Bound [11]) .

$$A(n, w, \lambda) \leq \left\lfloor \frac{n}{w} \left\lfloor \frac{(n-1)}{w-1} \left\lfloor \dots \left\lfloor \frac{(n-\lambda)}{w-\lambda} \right\rfloor \dots \right\rfloor \right\rfloor \right\rfloor .$$

If $w^2 - n\lambda > 0$ then

$$A(n, w, \lambda) \leq \left\lfloor \frac{n(w-\lambda)}{w^2 - n\lambda} \right\rfloor .$$

Continuing with the binary case, Agrell *et. al.* [12] establish the following bound.

Theorem 2 ([12]) .

$$A(n, w, \lambda) \leq n \text{ if } 0 < w^2 - n\lambda \leq w - \lambda$$

By identifying alphabet elements with mutually distinct binary strings of length m and weight at most one, an $(n, w, \lambda)_{m+1}$ code can be considered an (nm, w, λ) -code. As such the bounds on binary codes can easily be adapted to the non-binary case. Moreover, observe that an $(n, w, \lambda)_{m+1}$ code attaining the bound $A(n, w, \lambda)_{m+1}$ must have a coordinate in which at least $\frac{w \cdot A(n, w, \lambda)_{m+1}}{mn}$ codewords have a common nonzero entry. As observed in [13], shortening the code with respect to this coordinate gives a code with at most $A(n-1, w-1, \lambda-1)_{m+1}$ codewords.

Theorem 3 ([13])

$$A(n, w, \lambda)_{m+1} \leq \left\lfloor \frac{mn}{w} A(n-1, w-1, \lambda-1)_{m+1} \right\rfloor$$

Observing that $A(n, w, 0)_{m+1} = m \left\lfloor \frac{n-\lambda}{w-\lambda} \right\rfloor$, Theorems 1, 2, and 3 then give the following.

Theorem 4 (Johnson Bound Non-binary) .

$$A(n, w, \lambda)_{m+1} \leq \left\lfloor \frac{mn}{w} \left\lfloor \frac{m(n-1)}{w-1} \left\lfloor \dots \left\lfloor \frac{m(n-\lambda)}{w-\lambda} \right\rfloor \dots \right\rfloor \right\rfloor \right\rfloor .$$

If $w^2 > mn\lambda$ then

$$A(n, w, \lambda)_{m+1} \leq \min \left\{ mn, \left\lfloor \frac{mn(w-\lambda)}{w^2 - mn\lambda} \right\rfloor \right\} .$$

We note that the first bound in Theorem 4 may also be found in [14] with a proof (quite different from that given here) in [15].

Observe that by choosing a fixed linear ordering, each codeword from an $(\Lambda \times S \times T, w, \lambda)$ 3D-OOC C can be viewed as a binary constant weight (w) code of length ΛST . Moreover, by including the T distinct cyclic shifts of each codeword we obtain a corresponding constant weight binary code of size $T \cdot |C|$. It follows that

$$|C| \leq \left\lfloor \frac{A(\Lambda ST, w, \lambda)}{T} \right\rfloor \quad (1)$$

From the equation (1) and Theorem 4 we obtain the following bounds for 3-D OOCs.

Theorem 5 (Johnson Bound for 3D OOCs [16])

Let C be a $(\Lambda \times S \times T, w, \lambda)$ -OOC. Then

$$\Phi(C) \leq \left\lfloor \frac{\Lambda S}{w} \left\lfloor \frac{\Lambda ST - 1}{w - 1} \left\lfloor \dots \left\lfloor \frac{\Lambda ST - \lambda}{w - \lambda} \right\rfloor \dots \right\rfloor \right\rfloor \right\rfloor. \quad (2)$$

If $w^2 > \Lambda ST \lambda$ then

$$\Phi(C) \leq \min \left\{ \Lambda S, \left\lfloor \frac{\Lambda S(w - \lambda)}{w^2 - \Lambda ST \lambda} \right\rfloor \right\}. \quad (3)$$

We note that the first bound (2) may also found in [17].

Specializing now to ideal codes we observe that a $(\Lambda \times S \times T, w, 0, \lambda)$ 3D-OOC C can be viewed as a constant weight (w) code of length ΛS over an alphabet of size $T + 1$ containing zero (See Fig. 1 (a), (b)). By including the T distinct cyclic shifts of each codeword we obtain a corresponding constant weight code of size $T \cdot |C|$.

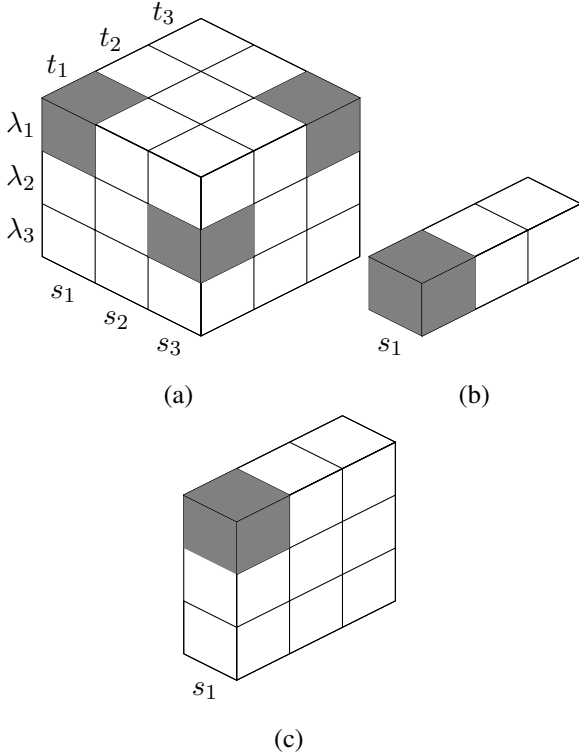


Figure 1: (a) A codeword from an ideal 3-D OOC, black cubes indicate 1, white indicate 0. (b) Each of the ΛS space/wavelength sections correspond to a (possibly zero) element from an alphabet of size $T + 1$. (c) If the code is AMOPP, then each of the S spatial planes correspond to a (possibly zero) element from an alphabet of size $\Lambda T + 1$.

It follows that

$$|C| \leq \left\lfloor \frac{A(\Lambda S, w, \lambda)_{T+1}}{T} \right\rfloor. \quad (4)$$

From Theorem 4 and the equation (4) we obtain the following bound for ideal 3-D OOCs.

Theorem 6 [Johnson Bound for Ideal 3D OOC]

Let C be an $(\Lambda \times S \times T, w, 0, \lambda)$ -OOC, then

$$\begin{aligned} \Phi(C) &\leq J(\Lambda \times S \times T, w, 0, \lambda_c) \\ &= \left\lfloor \frac{\Lambda S}{w} \left\lfloor \frac{T(\Lambda S - 1)}{w - 1} \left\lfloor \dots \left\lfloor \frac{T(\Lambda S - \lambda)}{w - \lambda} \right\rfloor \dots \right\rfloor \right\rfloor \right\rfloor \end{aligned}$$

Note that from Theorem 6 we see that if C is an ideal 3D OOC of maximal weight ($w = \Lambda S$) then $\Phi(C) \leq T^\lambda$

Similarly, (Fig. 1 (c)) an AMOPP OOC corresponds to a constant weight code of length S over an alphabet of size $\Lambda T + 1$ (containing zero). Consequently we obtain the following bound on AMOPP codes. This bound is also found (with a different proof) in [18], and in [16].

Theorem 7 [Johnson Bound for AMOPP OOC]

Let C be an $(\Lambda \times S \times T, w, 0, \lambda)$ -AMOPP OOC, then

$$\Phi(C) \leq \left\lfloor \frac{1}{T} \left\lfloor \frac{\Lambda ST}{w} \left\lfloor \frac{\Lambda T(S - 1)}{w - 1} \left\lfloor \dots \left\lfloor \frac{\Lambda T(S - \lambda)}{w - \lambda} \right\rfloor \dots \right\rfloor \right\rfloor \right\rfloor \right\rfloor$$

From the above theorem, we see that if C is an SPP code (an AMOPP code of maximal weight S) then $|C| \leq \Lambda^\lambda T^{\lambda-1}$.

As observed in [16], similar reasoning also gives the following two Theorems

Theorem 8 [Johnson Bound for AMOPW OOC]

Let C be an $(\Lambda \times S \times T, w, 0, \lambda)$ -AMOPW OOC, then

$$\Phi(C) \leq \left\lfloor \frac{1}{T} \left\lfloor \frac{\Lambda ST}{w} \left\lfloor \frac{ST(\Lambda - 1)}{w - 1} \left\lfloor \dots \left\lfloor \frac{ST(\Lambda - \lambda)}{w - \lambda} \right\rfloor \dots \right\rfloor \right\rfloor \right\rfloor \right\rfloor$$

Theorem 9 [Johnson Bound for AMOPT OOC]

Let C be an $(\Lambda \times S \times T, w, 0, \lambda)$ -AMOPT OOC, then

$$\Phi(C) \leq \left\lfloor \frac{1}{T} \left\lfloor \frac{\Lambda ST}{w} \left\lfloor \frac{\Lambda S(T - 1)}{w - 1} \left\lfloor \dots \left\lfloor \frac{\Lambda S(T - \lambda)}{w - \lambda} \right\rfloor \dots \right\rfloor \right\rfloor \right\rfloor \right\rfloor$$

Codes meeting the bounds in Theorems 5 - 9 will be said to be J -optimal. At present, constructions of infinite families of optimal ideal 3D OOCs are relatively scarce. The codes appearing in the literature seem to be exclusively of the AMOPP or SPP type. According to the bounds established above, it would seem that for comparable dimensions and weight it may be possible to construct ideal codes with larger capacity than the AMOPP or AMOPW codes. This is indeed the case. In the following sections we will provide constructions of codes meeting the bounds in Theorem 6. Table 1 will perhaps serve place our constructions in context.

Table 1: Summary of known constructions of families of optimal ideal 3D OOC. Unless stated otherwise, $\lambda_c = 1$.

$$p \text{ a prime, } q \text{ a prime power, } \theta(k, q) = \frac{q^{k+1}-1}{q-1}$$

Conditions	Constraint	Reference
$w = S \leq p$ for all p dividing ΛT	SPP	[19]
$w = S = \Lambda = T = p$	SPP	[20]
$w = S = 4 \leq \Lambda = q$, $T \geq 2$	SPP	[20]
$w = S = q + 1$, $\Lambda = q > 3, T = p > q$	SPP	[20]
$w = S = 3$ $\Lambda \equiv T \pmod{2}$	SPP	[18]
$w = 3, \Lambda T(S - 1)$ even, $\Lambda T(S - 1)S \equiv 0 \pmod{3}$, and $S \equiv 0, 1 \pmod{4}$ if $T \equiv 2 \pmod{4}$ and Λ is odd.	AMOPP	[18]
$w = q + 1, T = \theta(d, q)$, $\Lambda S = \theta(m - 1, q^{d+1})$, $d > 0, m > 1$		Th'm 14
$w = q^{d+1}, T = q - 1$, $\Lambda ST = q^{2d+2} - 1$,		Th'm 15

2 Preliminaries

Our techniques will rely heavily on the properties of finite projective and affine spaces. Such techniques have been used successfully in the construction of infinite families of optimal OOCs (for 1-D codes see [21, 22, 23, 24, 25], for 2-D codes see [6, 26]) We start with a brief overview of the necessary concepts. By $PG(k, q)$ we denote the classical (or Desarguesian) finite projective geometry of dimension k and order q . $PG(k, q)$ may be modeled with the affine (vector) space $AG(k + 1, q)$ of dimension $k + 1$ over the finite field $GF(q)$. Under this model, points of $PG(k, q)$ correspond to 1-dimensional subspaces of $AG(k, q)$, projective lines correspond to 2-dimensional affine subspaces, and so on. A d -flat Π in $PG(k, q)$ is a subspace isomorphic to $PG(d, q)$; if $d = k - 1$, the subspace Π is called a *hyperplane*. Elementary counting shows that the number of d -flats in $PG(k, q)$ is given by the Gaussian coefficient

$$\left[\begin{matrix} k + 1 \\ d + 1 \end{matrix} \right]_q = \frac{(q^{k+1} - 1)(q^{k+1} - q) \dots (q^{k+1} - q^d)}{(q^{d+1} - 1)(q^{d+1} - q) \dots (q^{d+1} - q^d)} \quad (5)$$

In particular the number of points of $PG(k, q)$ is given by $\theta(k, q) = \frac{q^{k+1}-1}{q-1}$. We will use $\theta(k)$ to represent this number when q is understood to be the order of the field. Further, we shall denote by $\mathcal{L}(k)$ the number of lines in $PG(k, q)$. For a point set A in $PG(k, q)$ we shall denote by $\langle A \rangle$ the span of A , so $\langle A \rangle = PG(t, q)$ for some $t \leq k$.

A *Singer group* of $PG(k, q)$ is a cyclic group of automorphisms acting sharply transitively on the points. The generator of such a group is known as a *Singer cycle*. Singer groups are known to exist in classical projective spaces of any order and dimension and their existence follows from that of primitive elements in a finite field.

In the sequel we make use of a Singer group that is most easily understood by modelling a finite projective space using a finite field. If we let β be a primitive element of $GF(q^{k+1})$, the points of $\Sigma = PG(k, q)$ can be represented by the field elements $\beta^0 = 1, \beta, \beta^2, \dots, \beta^{n-1}$ where $n = \theta(k)$. The non-zero elements of $GF(q^{k+1})$ form a cyclic group under multiplication. It is not hard to show that multiplication by β induces an automorphism, or collineation, on the associated projective space $PG(k, q)$ (see e.g. [27]). Denote by ϕ the collineation of Σ defined by $\beta^i \mapsto \beta^{i+1}$. The map ϕ clearly acts sharply transitively on the points of Σ .

We can construct 3-D codewords by considering orbits under subgroups of G . Let $n = \theta(k) = \Lambda \cdot S \cdot T$ where G is the Singer group of $\Sigma = PG(k, q)$. Since G is cyclic there exists a unique subgroup H of order T (H is the subgroup with generator $\phi^{\Lambda S}$).

Definition 10 (Projective Incidence Array)

Let Λ, S, T be positive integers such that $n = \theta(k) = \Lambda \cdot S \cdot T$. For an arbitrary pointset \mathcal{A} in $\Sigma = PG(k, q)$ we define the $\Lambda \times S \times T$ incidence array $A = (a_{i,j,k})$, $0 \leq i \leq \Lambda - 1$, $0 \leq j \leq S - 1$, $0 \leq k \leq T - 1$ where $a_{i,j,k} = 1$ if and only if the point corresponding to $\beta^{i+j \cdot \Lambda + k \cdot S \Lambda}$ is in \mathcal{A} .

If \mathcal{A} is a pointset of Σ with corresponding $\Lambda \times S \times T$ incidence array A of weight w , then $\phi^{\Lambda S}$ induces a cyclic shift on the temporal planes of A . For any such set \mathcal{A} , consider its orbit $Orb_H(\mathcal{A})$ under the group H generated by $\phi^{\Lambda S}$. The set \mathcal{A} has *full H -orbit* if $|Orb_H(\mathcal{A})| = T = \frac{n}{\Lambda S}$ and *short H -orbit* otherwise. If \mathcal{A} has full H -orbit then a representative member of the orbit and corresponding 3-D codeword is chosen. The collection of all such codewords gives rise to a $(\Lambda \times S \times T, w, \lambda_a, \lambda_c)$ -3D-OOC, where

$$\lambda_a = \max_{0 \leq i < j \leq T-1} \{ |\phi^{\Lambda S \cdot i}(\mathcal{A}) \cap \phi^{\Lambda S \cdot j}(\mathcal{A})| \} \quad (6)$$

and

$$\lambda_c = \max_{0 \leq i, j \leq T-1} \{ |\phi^{\Lambda S \cdot i}(\mathcal{A}) \cap \phi^{\Lambda S \cdot j}(\mathcal{A}')| \} \quad (7)$$

ranging over all $\mathcal{A}, \mathcal{A}'$ with full H -orbit.

2.1 An affine analogue of the Singer automorphism

A further automorphism of $\Sigma = PG(k, q)$ shall play a role in our constructions. It may be viewed as an affine analogue of the Singer automorphism. If a hyperplane Π_∞ (at infinity) is removed from $PG(k, q)$, what remains is $AG(k, q)$ -the k -dimensional affine space. One way to model $AG(k, q)$ is to view the points as the elements of $GF(q^k)$. Recall that the set $GF(q^k)^*$ of non-zero elements of $GF(q^k)$ forms a cyclic group under multiplication. Take α to be a primitive element (generator) of $GF(q^k)^*$. Each nonzero affine point corresponds in the natural way to α^j for some j , $0 \leq j \leq q^k - 2$. Denote by ψ the mapping of $AG(k, q)$ defined by $\psi(\alpha^j) = \alpha^{j+1}$ and $\psi(0) = 0$. The map ψ is an automorphism of $AG(k, q)$ and, moreover, ψ admits a natural extension to an automorphism $\hat{\psi}$ of $PG(k, q)$. Denote by \hat{G} the group generated by $\hat{\psi}$. The fundamental properties of the group \hat{G} central to the constructions here are (for details, see e.g. [28] [27].):

1. \hat{G} fixes the point P_0 corresponding to the field element 0, and acts sharply transitively on the $q^k - 1$ nonzero affine points of $PG(k, q)$.
2. \hat{G} acts cyclically transitively on the points of Π_∞ . In particular the subgroup $H = \langle \hat{\psi}^{\theta(k-1)} \rangle$ fixes Π_∞ pointwise.

The 3D-OOCs constructed using affine pointsets will therefore consist of codewords of dimension $\Lambda \times S \times T$, where $\Lambda \cdot S \cdot T = q^k - 1$.

Definition 11 (Affine Incidence Array) *Let Λ, S, T be positive integers such that $q^k - 1 = \Lambda \cdot S \cdot T$. For an arbitrary pointset \mathcal{A} in $AG(k, q)$ we define the $\Lambda \times S \times T$ incidence array $A = (a_{i,j,k}), 0 \leq i \leq \Lambda - 1, 0 \leq j \leq S - 1, 0 \leq k \leq T - 1$ where $a_{i,j,k} = 1$ if and only if the point corresponding to $\alpha^{i+\Lambda j+S\Lambda k}$ is in \mathcal{A} .*

If \mathcal{A} is a set of w nonzero affine points with corresponding $\Lambda \times S \times T$ incidence array A of weight w , then $\hat{\psi}^{\Lambda S}$ induces a cyclic shift on the temporal planes of A . For any such set \mathcal{A} , consider its orbit $Orb_{\hat{H}}(\mathcal{A})$ under the group $\hat{H} = \langle \hat{\psi}^{\Lambda S} \rangle$. If \mathcal{A} has full \hat{H} -orbit then a representative member of the orbit and

corresponding 3-dimensional codeword (say c) is chosen. The collection of all such codewords give rise to a $(\Lambda \times S \times T, w, \lambda_a, \lambda_c)$ -3D-OOC, where

$$\lambda_a = \max_{0 \leq i < j \leq T-1} \{ |\hat{\psi}^{S\Lambda \cdot i}(\mathcal{A}) \cap \hat{\psi}^{S\Lambda \cdot j}(\mathcal{A})| \} \quad (8)$$

and

$$\lambda_c = \max_{0 \leq i, j \leq T-1} \{ |\hat{\psi}^{S\Lambda \cdot i}(\mathcal{A}) \cap \hat{\psi}^{S\Lambda \cdot j}(\mathcal{A}')| \} \quad (9)$$

ranging over all $\mathcal{A}, \mathcal{A}'$ with full \hat{H} -orbit.

3 Optimal Ideal codes

3.1 Codes from projective lines, $\lambda_c = 1$

Let $\Sigma = PG(k, q)$ where $G = \langle \phi \rangle$ is the Singer group of Σ as in the previous section. Our work will rely on the following results about orbits of flats.

Theorem 12 (Rao [27], Drudge[29]) *In*

$\Sigma = PG(k, q)$, there exists a short G -orbit of d -flats if and only if $\gcd(k + 1, d + 1) \neq 1$. In the case that $d + 1$ divides $k + 1$ there is a short orbit \mathcal{S} which partitions the points of Σ (i.e. constitutes a d -spread of Σ). There is precisely one such orbit, and the G -stabilizer of any $\Pi \in \mathcal{S}$ is $Stab_G(\Pi) = \langle \phi^{\frac{\theta(k)}{\theta(d)}} \rangle$.

Let $\Sigma = PG(k, q)$, k odd with Singer group $G = \langle \phi \rangle$. Let \mathcal{S} be the line spread determined (as in Theorem 12) by G where say $Stab_G(\mathcal{S}) = H$. Consider a line $\ell \notin \mathcal{S}$. ℓ is incident with precisely $q + 1$ members of \mathcal{S} and H acts sharply transitively on the points of each line of \mathcal{S} , so ℓ is of full H -orbit, that is $|Orb_H(\ell)| = q + 1$, and the lines in $Orb_H(\ell)$ are disjoint. It follows that the number of full H -orbits of lines is

$$\begin{aligned} \# \text{ orbits} &= \frac{\mathcal{L}(k) - |\mathcal{S}|}{q + 1} \\ &= \frac{1}{q + 1} \cdot \left[\frac{(q^{k+1} - 1)(q^{k+1} - q)}{(q^2 - 1)(q^2 - q)} - \frac{\theta(k)}{q + 1} \right] \\ &= \frac{q \cdot \theta(k) \cdot \theta(k - 2)}{(q + 1)^2} \end{aligned} \quad (10)$$

For each full H -orbit of lines, select a representative member and corresponding (projective) $\Lambda \times S \times q + 1$ 3-D incidence array (codeword) where $\Lambda S = \frac{\theta(k)}{q+1}$ are fixed positive integers. The collection of all such codewords comprises a $(\Lambda \times S \times (q+1), q+1, \lambda_a, \lambda_c)$ -3DOOC C . As two lines intersect in at most

one point we have (Equation (7)) $\lambda_c = 1$. Moreover, since the lines in any particular full H -orbit $Orb_H(\ell)$ are disjoint, we have (Equation 6) $\lambda_a = 0$. Hence, C is a $(\Lambda \times T \times (q + 1), q + 1, 0, 1)$ -OOC. From the bound (Theorem 6) we have

$$\begin{aligned} \Phi(C) &= \Phi(\Lambda \times S \times (q + 1), q + 1, 0, 1) \\ &\leq \left\lfloor \frac{\frac{\theta(k)}{q+1}}{q+1} \left\lfloor \frac{(q+1)\left(\frac{\theta(k)}{q+1} - 1\right)}{q} \right\rfloor \right\rfloor \\ &= \frac{\theta(k) \cdot q^2 \cdot \theta(k-2)}{q(q+1)^2} \end{aligned} \quad (11)$$

Comparing (10) and (11) we see that C is in fact optimal. Noting that $\frac{\theta(k)}{q+1} = \theta\left(\frac{k-1}{2}, q^2\right)$, we have shown the following as in [16].

Theorem 13 *Let q be a prime power and let $t \geq 1$. For any factorisation $\Lambda S = \theta(t, q^2)$ There exists a J -optimal $(\Lambda \times S \times (q + 1), q + 1, 0, 1)$ -OOC.*

In the codes constructed in Theorem 13, codewords correspond to lines of $\Sigma = PG(k, q)$ not contained in a particular line-spread. In an analogous way we may generalize whereby codewords correspond to lines that are not contained in any element of a d -spread of Σ . We describe this construction as follows. Choose $d \geq 1, m > 1$ such that $k + 1 = m(d + 1)$. Let $G = \langle \phi \rangle$ be the Singer group as above, and let S be the d -spread determined (as in Theorem 12) by G where say $Stab_G(S) = H = \langle \phi^t \rangle$ where $t = \frac{\theta(k)}{\theta(d)}$. Let $\Lambda S = t$ be any integral factorization. Let ℓ be a line not contained in any spread element (a d -flat in S), and let A be the $\Lambda \times S \times \theta(d)$ projective incidence array corresponding to ℓ . As above, ℓ has a full H -orbit. Moreover, as H acts sharply transitively on the points of each spread element, it follows that A , when considered as a $\Lambda \times S \times \theta(d)$ codeword, satisfies $\lambda_a = 0$. For each such line ℓ we choose a representative element of it's H -orbit and include it's corresponding incidence array as a codeword. The aggregate of these codewords gives an ideal $(\Lambda \times S \times \theta(d), q + 1, 0, 1)$ -3D OOC, C . Let us now determine the capacity of C . Elementary counting shows

$$\mathcal{L}(k) = \frac{\theta(k)\theta(k-1)}{q+1}$$

We now have

$$\begin{aligned} |C| &= \frac{\mathcal{L}(k) - \mathcal{L}(d) \cdot \frac{\theta(k)}{\theta(d)}}{\theta(d)} \\ &= \frac{\theta(k)\theta(k-1)}{\theta(d)(q+1)} - \frac{\theta(d-1)\theta(k)}{\theta(d)(q+1)} \\ &= \frac{\theta(k)}{\theta(d)(q+1)} [\theta(k-1) - \theta(d-1)] \end{aligned} \quad (12)$$

From Theorem 6 we have the corresponding Johnson Bound is

$$\begin{aligned} \Phi(C) &\leq \left\lfloor \frac{\frac{\theta(k)}{\theta(d)}}{q+1} \left\lfloor \frac{\theta(d)\left(\frac{\theta(k)}{\theta(d)} - 1\right)}{q} \right\rfloor \right\rfloor \\ &= \frac{\theta(k)}{\theta(d)(q+1)} \left(\frac{\theta(k) - \theta(d)}{q} \right) \\ &= \frac{\theta(k)}{\theta(d)(q+1)} [\theta(k-1) - \theta(d-1)] \end{aligned} \quad (13)$$

Comparing (12) and (13) we see the codes obtained are J -optimal. With the observation that $\frac{\theta(k)}{\theta(d)} = \theta(m-1, q^{d+1})$, we have shown the following [16].

Theorem 14 *For $d \geq 1, m > 1$, and $\Lambda S = \theta(m-1, q^{d+1})$, there exists a J -optimal $(\Lambda \times S \times \theta(d), q + 1, 0, 1)$ -OOC.*

3.2 Ideal Codes from Affine Flats, $\lambda_c = 1$

In this subsection we will establish the truth of the following theorem.

Theorem 15 *For q a prime power, for each $d \geq 0$, and for any factorisation $\Lambda S = \theta(2d + 1)$ there exists an optimal $(\Lambda \times S \times q - 1, q^{d+1}, 0, 1)$ -OOC.*

Let $\Sigma = PG(k, q)$ where $E = \Sigma \setminus \Pi_\infty$ is the associated affine space $AG(k, q)$. Let $\hat{G} = \langle \hat{\psi} \rangle$ be the map as described in Section 2.1 based on the primitive element α of $GF(q^k)^*$. Our affine analog of Theorem 12 follows from Theorem 8 of [27].

Theorem 16 (Rao [27]) *A d -flat Π in $PG(k, q)$ is of full \hat{G} -orbit if and only if the origin $P_0 \notin \Pi$ and Π is not a subset of Π_∞ .*

With notation as above, let $k = 2d + 1, \Sigma = PG(k+1, q), AG(k+1, q) = E = \Sigma \setminus \Pi_\infty, \hat{G} = \langle \hat{\psi} \rangle$, and $\hat{H} = \langle \hat{\psi}^{\theta(k-1)} \rangle$. Let S be a d -spread of Π_∞ , where say $S = \{S_1, S_2, \dots, S_{q^{d+1}+1}\}$.

Each affine $(d + 1)$ -flat Γ extends to a projective $(d + 1)$ -flat $\hat{\Gamma}$ of Σ which must meet Π_∞ in a d -flat.

Simple counting shows each S_i to be incident with precisely $q^{d+1} - 1$ of these $(d + 1)$ -flats which do not contain the affine zero point. Denote the aggregate of these projective (resp. affine) $(d + 1)$ -flats by $\hat{\Gamma}_i$ (resp. Γ_i), $1 \leq i \leq q^{k+1} - 1$. Under the action of \hat{H} , each $\hat{\Gamma}_i$ has full orbit (Theorem 16), so in particular there are $\theta(k)$ distinct orbits of the $\hat{\Gamma}_i$'s.

Let $\Lambda S = \theta(k)$, and $T = q - 1$. For each (full) \hat{H} orbit of Γ_i 's select a representative member and a corresponding $\Lambda \times S \times T$ affine incidence array (codeword). Let the aggregate of these codewords of weight q^{d+1} be the code C . To prove Theorem 15 it remains to show $\lambda_a = 0$, $\lambda_c = 1$, and that C is optimal.

We claim that $|\Gamma_i \cap \Gamma_j| \leq 1$ for $i \neq j$. Indeed, if two points $P, Q \in \Gamma_i \cap \Gamma_j$, then the projective line $\langle P, Q \rangle$ meets Π_∞ in a point of $\hat{\Gamma}_i \cap \hat{\Gamma}_j$. As \mathcal{S} is a spread it follows that $\hat{\Gamma}_i \cap \Pi_\infty = \hat{\Gamma}_j \cap \Pi_\infty$ and therefore $\Gamma_i = \Gamma_j$. As a result we have $\lambda_c = 1$. Likewise, a dimension argument shows $\lambda_a = 0$. Optimality follows from the second bound in Theorem 5.

4 Conclusion

In this paper we provided constructions of infinite families of 3-dimensional OOC's. In each case the families have ideal autocorrelation $\lambda_a = 0$ and are optimal. A key feature of the constructions presented involve two or more parameters that may grow without bound and at each stage produce optimal codes.

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