DESIGN OF NONLINEAR OPTIMAL CONTROL SYSTEMS USING JORDAN CONTROLLED FORM

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Abstract: - The problem of control systems design for nonlinear plants still has no exhaustive solution. This problem is commonly solved on the basis of transformation of the nonlinear plant equations to some simple. Such approach simplifies the solution of the control system design problem and makes the solution analytical. For design of the optimal control systems for the nonlinear plants, their equations are expedient to transform to Jordan controlled form (JCF). The analytical design method of the optimal control systems for nonlinear plants with using the JCF of their equations is proposed in this paper. This problem has a solution if all plant state variables are measured. The JCF exists, if the nonlinear plant is completely controllable. The proposed method includes two steps. At the first step, a linearization control is designed by a nonlinear reversible transformation of the plant state variables. Under the linearization control, the system equations are linear and stationary in the new variables. Theorem about existence of a linearization control is proved. At the second step, the optimal control is designed as optimal in the sense of a minimum of nonlinear quadratic criteria. This control is designed using solution of the Riccati equation. Optimality of the obtained nonlinear control is proved also. Design of the optimal control systems for nonlinear plants using JCF is expedient, because the equations of many real plants have JCF or can be easily transformed to this form. Frequently, the plant equations convert into this form if the state variables are designated in appropriate way. The example of the optimal control system design for nonlinear plant is given.

Key-Words: - Nonlinear plant, reversible transformation, Jordan controlled form, linearization control, optimal control, nonlinear quadratic criteria.

1 Introduction

The method of the plant equations transformation to some simple form is widely used for solution of the design problem of the nonlinear control systems. This approach simplifies the solution of the design problem and makes it analytical. The choice of the suitable form to which the plant equations transformation is more convenient is the basic difficulty of this approach. Corresponding methods for transformation of the equations are known usually. However, when the plant equations are in general form, the process of their transformation to the chosen form is rather difficult [1 – 5].

In nonlinear cases, the equations are transformed to the normal canonical control form [1 – 3], triangular form [4, 5], Lukyanov-Utkin regular form [6], Jordan controlled form [7, 8], and others. If equations of the plant are represented in the triangular form, the backstepping method to design an adaptive control system is applied very easily [4, 5]. The Lukyanov-Utkin regular form of the plant equation allows decoupling the initial problem of a control system design on several tasks of the smaller dimension [6]. If the equations of a nonlinear plant are converted to JCF, then the control providing stability of the system equilibrium or full compensation of influence of the bounded external disturbances is designed easily [7, 8].

Optimal control systems also can be designed using JCF. Usually the optimal control systems for the nonlinear plants are designed with application of the first approximation equations [9] or an approximate solution of the nonlinear differential Hamilton-Jacobi-Bellman equation [3, 9]. However, in this case, the basic advantages of the optimal approach – analyticity and simplicity are lost and control received by such way is not optimal.

The optimal control systems for nonlinear plants are designed by the transformation method usually in two steps. At the first step a linearization control is designed. Optimal control is designed at the second step [10].

In this paper the analytical design method of the optimal control systems for the nonlinear plants, whose equations are converted to JCF, is considered. The optimal control is designed by this method in two steps also. The linearization control is constructed analytically on the basis of the stabilizing control which was proposed in [7]. Optimal control in the sense of a minimum of nonlinear quadratic criteria is designed at the second step with using solution of the Riccati equation [1, 11]. Therefore, the design procedure of the nonlinear optimal control systems is analytical. Desirable character of the system transient processes is provided by the appropriate choice of the nonlinear quadratic criteria coefficients as in the linear case [12]. The proposed method is important for practice, because the equations of many real plants have JCF or can be converted to this form by change of the state variables designations or by a nonlinear transformation [7, 12].

The paper is organized as follows. In section 2 the mathematical definition of the JCF of plants equations and formulation of the considered problem are given. The proposed method of the optimal control systems design is considered in section 3. In subsection 3.1 the design method of a linearization control is considered. The basic paper result is the method of the optimal control system design is stated in subsection 3.2. The theorem about optimality of the received control is proved here. An example of the optimal control system design for nonlinear plant is given in subsection 3.3. The concluding remarks are given in the last section.

2 JCF and Problem Formulation

Suppose some plant with a single control input is described by the equation

$$\dot{x} = f(x) + e_n u_0, \tag{1}$$

where $x \in R^n$ is the state vector; $f(x) = [f_1(\bar{x}_2)\ f_2(\bar{x}_3)\ \dots f_{n-1}(\bar{x}_n)\ f_n(\bar{x}_n)]^T$ is the nonlinear vector-function; $f_i(\bar{x}_{i+1})$ is an scalar, continuous function which is differentiable $n-i$ times on all its arguments; $\bar{x}_i = [x_1 \dots x_i]^T$ is a sub vector including the first *i* state variables $x_1, \dots, x_i$; evidently $\bar{x}_n = x$; e_n is *n*-th column of an identity $n \times n$ -matrix; $u_0 = u_0(x)$ is the scalar control.

Let $x^\circ = x^\circ(t, u_0^\circ)$ be a vector that describes the unperturbed motion of the system (1); u_0° is an appropriate control.

Let's enter a vector of deviations $\tilde{x} = x - x^\circ$ and a deviation $u = u_0 - u_0^\circ$ of the control. For clarity, the equations of the system (1) in deviations are expressed in a scalar form:

$$\dot{\tilde{x}}_i = \phi_i(\tilde{x}_1, \dots, \tilde{x}_{i+1}), \quad i = \overline{1, n-1}, \tag{2}$$

$$\dot{\tilde{x}}_n = \phi_n(\tilde{x}_1, \dots, \tilde{x}_n) + u, \quad y = c^T \tilde{x}, \tag{3}$$

where $\tilde{x}_i = x_i - x_i^\circ$ is the deviation of the state variable x_i, $i = \overline{1, n}$; $\phi_i(\tilde{x}_1, \dots, \tilde{x}_{i+1}) = f_i(\bar{x}_{i+1}) - f_i(\bar{x}_{i+1}^\circ) = \phi_i(\bar{\tilde{x}}_{i+1})$, $i = \overline{1, n-1}$ and $\phi_n(\tilde{x}_1, \dots, \tilde{x}_n) = f_n(x) - f_n(x^\circ) = \phi_n(\tilde{x})$; $\bar{\tilde{x}}_i = [\tilde{x}_1 \dots \tilde{x}_i]^T$; $u = u(\tilde{x})$ is control action. The variables $\tilde{x}_i$, are measured and $\phi_i(0) = 0$, $i = \overline{1, n}$; $\bar{\tilde{x}}_n = \tilde{x}$ evidently.

The control design problem for equations (2), (3) has a solution only when for all $i = \overline{1, n-1}$,

$$\left| \frac{\partial \phi_i(\tilde{x}_1, \dots, \tilde{x}_{i+1})}{\partial \tilde{x}_{i+1}} \right| \ge \varepsilon \ne 0, \quad \tilde{x} \in \Omega_{\tilde{x}} \in R^n, \tag{4}$$

where ε there is any positive number; $\Omega_{\tilde{x}}$ is some domain of the space R^n. This domain should include an equilibrium $\tilde{x} = 0$.

Definition: If the equations (2), (3) satisfy conditions (4), they are called Jordan controlled form [7, 8, 12].

Evidently, the canonical Frobenius form of the system or plant equations is a special case of JCF, where $\phi_n(\tilde{x}) = -\alpha_0 \tilde{x}_1 - \alpha_1 \tilde{x}_2 - \dots - \alpha_{n-1} \tilde{x}_n$; $\phi_i(\tilde{x}) = \tilde{x}_{i+1}$, $i = \overline{1, n-1}$ (for $n > 1$) [2, 7].

The design problem consists in the definition of an optimal control $u = u_{opt}(\tilde{x})$ under which the uncertain nonlinear quadratic criteria J satisfies to the next condition

$$J = \int_0^\infty [\tilde{x}^T Q(\tilde{x}) \tilde{x} + \rho \gamma_1^2(\tilde{x})(u - u_{lin})^2] dt \to \min_u \tag{5}$$

for all $\tilde{x} \in \Omega_{\tilde{x}} \in R^n$. Here $Q(\tilde{x}) = S^T(\tilde{x}) \bar{Q} S(\tilde{x})$; $\bar{Q} \ge 0$ is a constant, symmetric, nonnegative matrix and $\rho > 0$ is a positive number.

The matrix $S(\tilde{x})$, the function $\gamma_1(\tilde{x})$ and the control $u_{lin}(\tilde{x})$ will be determined later. Values of the number ρ and coefficients of the matrix $\bar{Q}$ are chosen according to desirable character of the system transient process.

3 Problem Solution

The plant equations are supposedly submitted in JCF. Therefore the optimum control system is designed in two steps. At the first step, a linearization control is constructed on the basis of the stabilizing control and transformation of the received system equations to the new variables. The stabilizing control was proposed in [7]. Existence of the linearization control is caused by Jordan canonical form of the plant equations (2), (3).

Optimal control is designed at the second stage. The optimality of this control is proved by using an optimality condition from [13, p. 322]. Let's pass to implementation of this approach.

3.1 Linearization control design

To solve this problem the new state vector $w = w(\tilde{x}) = [w_1(\tilde{x}) \quad w_2(\tilde{x}) \quad \ldots \quad w_n(\tilde{x})]^T$ is determined as follows: $w_1 = \tilde{x}_1$,

$$w_i(\bar{\tilde{x}}_i) = \sum_{\nu=1}^{i-1} \frac{\partial w_{i-1}}{\partial \tilde{x}_\nu} \phi_\nu(\bar{\tilde{x}}_{\nu+1}) + \lambda_{i-1} w_{i-1}(\bar{\tilde{x}}_{i-1}), \quad (6)$$

$i = \overline{2, n}$; here and further λ_i is any constant, $i = \overline{1, n}$.

The transformation $w(\tilde{x})$ (6) is continuous, bounded and reversible by virtue of the conditions (4). As the functions $\phi_i(\bar{\tilde{x}}_{i+1})$, $i = \overline{1, n-1}$ are differentiable and $w(0) = 0$, the nonlinear transformation $w = w(\tilde{x})$ (6) in domain $\Omega_{\tilde{x}} \in R^n$ can be presented in the quasilinear form:

$$w = S(\tilde{x})\tilde{x}, \quad (7)$$

where $S(\tilde{x})$ is an $n \times n$-matrix and $\det S(\tilde{x}) \neq 0$ [12]. Therefore in domain $\Omega_{\tilde{x}} \in R^n$ there is a bounded, reverse transformation $\tilde{x} = S^{-1}(\tilde{x}) w(\tilde{x})$, where the vector $w(\tilde{x})$ is defined by expressions (6). At these conditions the stabilizing control u for the plant (2), (3) can be defined by the expression

$$u = -\gamma_1^{-1}(\tilde{x})\left[\gamma_2(\tilde{x}) + \lambda_n w_n(\tilde{x})\right] - \phi_n(\tilde{x}) + \gamma_1^{-1}(\tilde{x}) v, \quad (8)$$

where

$$\gamma_1(\tilde{x}) = \frac{\partial w_n(\tilde{x})}{\partial \tilde{x}_n} = \prod_{i=1}^{n-1} \frac{\partial \phi_i(\bar{\tilde{x}}_{i+1})}{\partial \tilde{x}_{i+1}}; \quad (9)$$

$$\gamma_2(\tilde{x}) = \sum_{i=1}^{n-1} \frac{\partial w_n(\tilde{x})}{\partial \tilde{x}_i} \tilde{\phi}_i(\bar{\tilde{x}}_{i+1}), \quad \tilde{x} \in \Omega_{\tilde{x}}; \quad (10)$$

ν is new control – some function of $\tilde{x}$ or time t.

Theorem 1: If in some domain $\Omega_{\tilde{x}}$ the condition (4) is satisfied and the control u is defined by the expressions (8) – (10), the equations of the system (2), (3), (8) concerning to vector w are linear and have the following kind:

$$\dot{w} = \Lambda_n w + e_n v, \quad (11)$$

where $e_n = [0 \quad \ldots \quad 0 \quad 1]^T$,

$$\Lambda_n = \begin{bmatrix} -\lambda_1 & 1 & \ldots & 0 \\ 0 & -\lambda_1 & \ddots & 0 \\ \vdots & \vdots & \ddots & 1 \\ 0 & 0 & \ldots & -\lambda_n \end{bmatrix}. \quad (12)$$

Proof: The expression (6) in the form of (2) where $i = 2, 3, \ldots, n-1$ can be presented as follows:

$$w_2 = \dot{w}_1 + \lambda_1 w_1,$$

$$w_3 = \dot{w}_2 + \lambda_2 w_2, \ldots, \quad w_n = \dot{w}_{n-1} + \lambda_{n-1} w_{n-1}. \quad (13)$$

Further, we will substitute control $u = u(\tilde{x}, \nu)$ (8) in the form of (10) into the equation (3) and we will multiply both its parts on $\gamma_1(\tilde{x})$. As result, we will obtain equation

$$\gamma_1(\tilde{x})\dot{\tilde{x}}_n = -\sum_{i=1}^{n-1} \frac{\partial w_n(\tilde{x})}{\partial \tilde{x}_i} \tilde{\phi}_i(\bar{\tilde{x}}_{i+1}) - \lambda_n w_n(\tilde{x}) + v. \quad (14)$$

According to the equation (9) $\gamma_1(\tilde{x}) = \partial w_n(\tilde{x}) / \partial \tilde{x}_n$, therefore the equation (14) can be copied so

$$\sum_{i=1}^{n-1} \frac{\partial w_n(\tilde{x})}{\partial \tilde{x}_i} \tilde{\phi}_i(\bar{\tilde{x}}_{i+1}) + \frac{\partial w_n(\tilde{x})}{\partial \tilde{x}_n} \dot{\tilde{x}}_n = -\lambda_n w_n(\tilde{x}) + v. \quad (15)$$

The left part of the equation (15), evidently, is the time derivative of the variable w_n, therefore it can be written down as follows: $\dot{w}_n = -\lambda_n w_n + v$. Now the statement of the theorem 1 follows from last equality and from the expressions (13). *Theorem 1 is proved*.

The matrix Λ_n (12), where $\lambda_i = -\lambda$, $i = \overline{1, n}$, evidently coincides with the Jordan cell of size $n \times n$ [14, p. 142]. Therefore, (2), (3) with (4) in some domain $\Omega_{\tilde{x}} \in R^n$ referred to as Jordan controlled form.

Evidently, the system (11), (12) is asymptotically stable if $\lambda_i \geq \varepsilon > 0$, $i = \overline{1, n}$. Since the transformation (7) is reversible, the equilibrium $\tilde{x} = 0$ of the system (2) – (4), (8) – (10) with $\lambda_i \geq \varepsilon > 0$, $i = \overline{1, n}$ is also asymptotically stable in the domain $\Omega_{\tilde{x}} \in R^n$. Therefore, control (8) – (10) under the mentioned above conditions is stabilizing.

However, the expressions (8) – (15) are carried out by all values λ_i, including $\lambda_i = 0$, $i = \overline{1, n}$. In this case, the nonlinear system (2), (3), (8) in variables w_i is described by the linear equations (11), (12) with $\lambda_i = 0$, $i = \overline{1, n}$. Hence for an nonlinear plant (2), (3), (4) the linearization control can be defined by the expression

$$u = u_{lin} + \gamma_1^{-1}(\tilde{x})v\,,\;\; u_{lin} = -\gamma_1^{-1}(\tilde{x})\gamma_2(\tilde{x}) - \phi_n(\tilde{x})\,, \quad (16)$$

where

$$w_1 = \tilde{x}_1\,,\;\; w_i(\bar{\tilde{x}}_i) = \sum_{\nu=1}^{i-1} \frac{\partial \bar{w}_{i-1}}{\partial \tilde{x}_\nu} \phi_\nu(\bar{\tilde{x}}_{\nu+1})\,,\;\; i = \overline{2,n}\,, \quad (17)$$

the functions $\gamma_1(\tilde{x})$, $\gamma_2(\tilde{x})$ are determined still by the expressions (9), (10).

The linearization control (16), (17), (9), (10) is used for design of the control which is optimal in the sense of a minimum of the nonlinear quadratic criteria *J* (5).

3.2 Optimal control system design

First of all, we stipulate that the nonlinear quadratic criterion *J* (5) is completely certain now, as the matrix $S(\tilde{x})$ and the function $\gamma_1(\tilde{x})$ are determined by expression (7) and by equality (9).

For solution of the optimization problems (2) – (5), the control $u = u(\tilde{x})$ is taken as (16). Equations (2), (3) with $u = u_{lin} + \gamma_1^{-1}(\tilde{x})v$ are given as follows:

$$\dot{\tilde{x}} = \phi_{lin}(\tilde{x}) + e_n v\,, \quad (18)$$

where $\phi_{lin}(\tilde{x}) = \phi(\tilde{x}) + e_n u_{lin}(\tilde{x})$.

According to the theorem 1, the equation (18) concerning the vector *w* (17) looks like:

$$\dot{w} = \bar{\Lambda}_n w + e_n v\,,\;\; \bar{\Lambda}_n = \begin{bmatrix} 0 & 1 & \dots & 0 \\ 0 & 0 & \ddots & 0 \\ \vdots & \vdots & \ddots & 1 \\ 0 & 0 & \dots & 0 \end{bmatrix}, \quad (19)$$

where $v = v(w) = v(\tilde{x})\big|_{\tilde{x}=S^{-1}(\tilde{x})w}$.

As the system (19) is linear and stationary, the control providing a minimum of quadratic criterion

$$\bar{J} = \int_0^\infty [w^T \bar{Q} w + \rho v^2(w)]\,dt \quad (20)$$

on trajectories of the system (19) is defined [11, p. 275] by the expression

$$v_{opt}(w) = -\rho^{-1} e_n^T P\, w\,. \quad (21)$$

Here *P* is the symmetric, positive matrix. It is a solution of the Riccati equation

$$P\bar{\Lambda}_n + \bar{\Lambda}_n^T P - P e_n \rho^{-1} e_n^T P + \bar{Q} = 0\,. \quad (22)$$

Let $\bar{Q}$ and ρ in the equation (22) are the matrix and the number from the nonlinear quadratic criteria *J* (5). Then the solution of the optimization problem (2), (3), (5) under the condition (4) is defined by the following theorem.

Theorem 2: Let the control $u_{lin} = u_{lin}(\tilde{x})$ and the function $\gamma_1(\tilde{x})$ are determined by the expressions (16), (9), the matrix $S(\tilde{x})$ is from equality (7); the matrix $\bar{Q}$ and the number ρ are from the nonlinear quadratic criteria *J* (5) and *P* is the matrix Riccati equation (22). Then the optimal control, delivering a minimum of the nonlinear quadratic criteria *J* (5) on trajectories of the closed nonlinear system (2), (3), (4), is defined by the expression

$$u_{opt} = u_{lin}(\tilde{x}) - \gamma_1^{-1}(\tilde{x})\rho^{-1} e_n^T P\, S(\tilde{x})\tilde{x}\,. \quad (23)$$

Proof: The system (19) with control $v = v_{opt}$ (21) is globally stable [11, p. 275]. The transformation $w = w(\tilde{x})$ (17), similarly to (7), is reversible, continuous and bounded by virtue of the condition (4). Therefore, the equilibrium $\tilde{x} = 0$ of the system (2), (3) with $u = u_{opt}$ (23) is asymptotically stable in the domain $\Omega_{\tilde{x}} \in R^n$. Hence, the integrals *J* (5) and $\bar{J}$ (20) can be written down as follows:

$$J = \int_0^{t_1} [\tilde{x}^T S^T(\tilde{x}) \bar{Q} S(\tilde{x})\tilde{x} + \rho\gamma_1^2(\tilde{x})(u - u_{lin})^2]\,dt\,, \quad (24)$$

$$\bar{J} = \int_0^{t_1} [w^T \bar{Q} w + \rho v^2]\,dt\,, \quad (25)$$

where t_1 there is a big enough number.

Statement [13, p. 322]: If the certain integral

$$J_1 = \int_0^{t_1} F(w, \bar{u})\,dt$$

has a minimum on an function $\bar{u}$, then there is a positive number ε_{in} such that

$$\Delta J_1 = \int_0^{t_1} \Delta F(w, \bar{u})\,dt = \int_0^{t_1} \frac{\partial F(w, \bar{u})}{\partial \bar{u}} \delta\bar{u}\,dt > 0\,, \quad (26)$$

under any variation $\delta\bar{u}$, which satisfies inequality

$$0 < |\delta\bar{u}| < \varepsilon_{in}\,. \quad (27)$$

Here $\delta\bar{u}$ is a variation of the function $\bar{u}$.

The partial derivative and variation, contained in (26), concerning to the integral (25) and function *v* are determined by expressions:

$$\frac{\partial F(w,v)}{\partial v} = 2\rho v = -2 e_n^T P\, w\,,\;\; \delta v = -\rho^{-1} e_n^T P \delta w\,,$$

where $\delta w = [\delta w_1 \;\; \delta w_2 \;\; \dots \;\; \delta w_n]^T$ is any variation of the vector *w*.

Integral (25) under the theorem 3.7 [11, p. 275] has a minimum by the control $v = v_{opt}(w)$ (21). Therefore, according to the inequalities (26), (27) there is a number $\overline{\varepsilon}_{in}$ such that the next inequalities are carried out

$$\Delta\overline{J} = \int_0^{t_1} 2e_n^T P w \cdot \rho^{-1} e_n^T P \delta w \, dt > 0 \,,$$

$$0 < \left| \rho^{-1} e_n^T P \delta w \right| < \overline{\varepsilon}_{in} \,. \quad (28)$$

Let $\Omega_{\tilde{x}0} \in \Omega_{\tilde{x}}$ be an attraction domain of the equilibrium of the stable system (2), (3), (23). Vectors $w(t)$ and $\tilde{x}(t)$ are bounded if vector $\tilde{x}(0) \in \Omega_{\tilde{x}0}$. The transformation $w = w(\tilde{x}) = S(\tilde{x})\tilde{x}$ is reversible and the matrix $S(\tilde{x})$ is bounded and differentiable on $\tilde{x}$ at domain $\Omega_{\tilde{x}}$. In these conditions, variations of the vectors w and $\tilde{x}$ are connected by expression $\delta w = \left[[\tilde{x}^T (S^T(\tilde{x}))']^T + S(\tilde{x}) \right] \delta\tilde{x}$ [13]. Here $(S^T(\tilde{x}))' = \partial S^T(\tilde{x}) / \partial \tilde{x}$ is $n \times n$-matrix; its (i,j)-element is the vector-line of a gradient $\partial S_{ji}(x) / \partial x$ and $\delta\tilde{x} = [\delta\tilde{x}_1 \quad \delta\tilde{x}_2 \quad \dots \quad \delta\tilde{x}_n]^T$ is a variation of the vector $\tilde{x}$.

Using expression $\delta w = \left[[x^T (S^T(x))']^T + S(x) \right] \delta x$ we will present the inequalities (28) as follows:

$$\Delta\overline{J} = \int_0^{t_1} \{-2e_n^T P S(\tilde{x})\tilde{x}\} \{-\rho^{-1} e_n^T P \Big[[\tilde{x}^T (S^T(\tilde{x}))']^T + $$
$$+ S(\tilde{x}) \Big] \delta\tilde{x}\} \, dt > 0 \,,$$

$$0 < \left| \rho^{-1} e_n^T P \left[[\tilde{x}^T (S^T(\tilde{x}))']^T + S(\tilde{x}) \right] \delta\tilde{x} \right| < \overline{\varepsilon}_{in} \,. \quad (29)$$

Let's show, that conditions (26), (27) of a minimum of the nonlinear quadratic criteria J (5) on trajectories of the system (2), (3), (23) are similar to inequalities (29). Really, the function $F(w, \overline{u})$ from the condition (26), (27) in case of the integral J (24) and control (23) looks like

$$F(w,u) = \tilde{x}^T S^T(\tilde{x}) Q S(\tilde{x})\tilde{x} + \rho\gamma_1^2(\tilde{x})(u - u_{lin})^2 \,.$$

Hence

$$\left.\frac{\partial F(w,u)}{\partial u}\right|_{u=u_{opt}} = 2\rho\gamma_1^2(\tilde{x})(u - u_{lin})\Big|_{u=u_{opt}} =$$

$$= -2\gamma_1(\tilde{x}) e_n^T P S(\tilde{x})\tilde{x} \,. \quad (30)$$

The components u_{lin} and $\gamma_1(\tilde{x})$ are determined unequivocally by the given equations (2), (3), therefore, the variation δu_{opt} of the optimal control (23) is defined by the expression

$$\delta u_{opt}(\tilde{x}) = -\gamma_1^{-1}(\tilde{x}) \rho^{-1} e_n^T P \left[[\tilde{x}^T (S^T(\tilde{x}))']^T + S(\tilde{x}) \right] \delta\tilde{x} \,. \quad (31)$$

If to substitute the expressions (30), (31) in the inequalities (26), (27), the minimum conditions of the integral (24), which is equivalent to the criteria J (5), on the trajectories of the system (2), (3), (23) will coincide with inequalities (29) completely. Last inequalities are carried out as shown above; hence the conditions of a minimum of the nonlinear quadratic criteria J (5) on the trajectories of the system (2), (3), (23) are carried out also. *The theorem 2 is proved.*

The received expressions (17), (9), (10), (7), (22), (23) are a theoretical basis of the proposed method of optimal control systems design for plants, which equations are submitted in JCF. This method is shown in the example below.

3.3 An example

Suppose, the plant is described by the equations

$$\dot{x}_1 = 2(x_2 + x_3^2) \,, \; \dot{x}_2 = u \,, \; \dot{x}_3 = (1 + x_3^2) x_1 \,. \quad (32)$$

It is necessary to find an optimal control by which the condition (5) is carried out with the number $\rho = 1$ and the matrix $\overline{Q} = \mathrm{diag}\{4 \quad 2 \quad 1\}$. The matrix $S(\tilde{x})$ and the function $\gamma_1(\tilde{x})$ are determined below.

The form of the equations (32), evidently, does not meet JCF. When the equations (32) would have JCF, designations of the state variables change so: $x_1 = \tilde{x}_2$, $x_2 = \tilde{x}_3$, $x_3 = \tilde{x}_1$. The resulting equations of the plant look like

$$\dot{\tilde{x}}_1 = (1 + \tilde{x}_1^2)\tilde{x}_2 = \phi_1(\tilde{x}) \,;$$

$$\dot{\tilde{x}}_2 = 2(\tilde{x}_3 + \tilde{x}_1^2) = \phi_2(\tilde{x}) \,; \quad \dot{\tilde{x}}_3 = u \,. \quad (33)$$

Equation (33) satisfy to the conditions (4), since $\partial\phi_1 / \partial\tilde{x}_2 = 1 + \tilde{x}_1^2$ and $\partial\phi_2 / \partial\tilde{x}_3 = 2$ for any $\tilde{x} \in R^3$, $\|\tilde{x}\| < \infty$. Therefore, these equations have JCF and the task has a solution. According to the proposed method, a linearization control is designed in beginning. For this purpose, the transformation (17) is determined for the equations (33):

$$w_1 = \tilde{x}_1 \,, \qquad w_2 = (1 + \tilde{x}_1^2)\tilde{x}_2 \,,$$
$$w_3 = 2(\tilde{x}_1 + \tilde{x}_1^3)\tilde{x}_1 + 2(1 + \tilde{x}_1^2)(\tilde{x}_1\tilde{x}_2^2 + \tilde{x}_3)$$

or in the vector-matrix form (7):

$$w(\tilde{x}) = S(\tilde{x})\tilde{x} \,, \quad (34)$$

where

$$S(\tilde{x}) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & (1 + \tilde{x}_1^2) & 0 \\ 2(\tilde{x}_1 + \tilde{x}_1^3) & \psi_{32}(\tilde{x}) & 2(1 + \tilde{x}_1^2) \end{bmatrix} . \quad (35)$$

Here the function $\psi_{32}(\tilde{x}) = 2(1+\tilde{x}_1^2)\tilde{x}_1\tilde{x}_2$. In this case $\det S(\tilde{x}) = 2(1+\tilde{x}_1^2)^2 \neq 0$. Therefore, the transformation (34), (35) is reversible and bounded for all $\tilde{x} \in R^3$, $\|\tilde{x}\| < \infty$. Functions $\gamma_1(\tilde{x})$ and $\gamma_2(\tilde{x})$ are defined by the expression (9), (10) and (33) as:

$$\gamma_1(\tilde{x}) = 2(1+\tilde{x}_1^2),$$
$$\gamma_2(\tilde{x}) = [\zeta_1(\tilde{x})\tilde{x}_2\tilde{x}_1 + \zeta_2(\tilde{x})](1+\tilde{x}_1^2), \quad (36)$$

where

$$\zeta_1(\tilde{x}) = 16\tilde{x}_1^2 + 6\tilde{x}_1\tilde{x}_2^2 + 12\tilde{x}_3,$$
$$\zeta_2(\tilde{x}) = 4\tilde{x}_1\tilde{x}_2 + 2\tilde{x}_2^3. \quad (37)$$

Now, according to the second expression (16), the linearization control will be written down as

$$u_{lin}(\tilde{x}) = -0,5[\zeta_1(\tilde{x})\tilde{x}_2\tilde{x}_1 + \zeta_2(\tilde{x})]. \quad (38)$$

Further, the control $u_{opt}(\tilde{x})$ is determined. Solution of the Riccati equation (22), where matrix $\bar{Q} = \mathrm{diag}\{4 \quad 2 \quad 1\}$ and $\rho = 1$, is

$$P = \begin{bmatrix} 7,368 & 5,785 & 2 \\ 5,785 & 8,656 & 3,684 \\ 2 & 3,684 & 2,893 \end{bmatrix}. \quad (39)$$

So, according to the expressions (23), (38), the function $\gamma_1(\tilde{x})$ (36), the matrices $S(\tilde{x})$ (35) and P (39) the optimal control, at which the condition (5) satisfies with $\bar{Q} = \mathrm{diag}\{4 \quad 2 \quad 1\}$ and $\rho = 1$, equals

$$u_{opt}(\tilde{x}) = u_{lin}(\tilde{x}) + v_{opt}(\tilde{x}), \quad (40)$$

where

$$v_{opt}(\tilde{x}) = -\tilde{x}_1(1+\tilde{x}_1^2)^{-1} - 2,893(\tilde{x}_1^2 + \tilde{x}_1\tilde{x}_2^2 + $$
$$+\tilde{x}_3) - 1,842\tilde{x}_2. \quad (41)$$

Plots of the state variable $\tilde{x}_1(t)$ and the control $u_{opt}(t)$ of the designed optimal system are shown on fig. 1 and fig. 2.

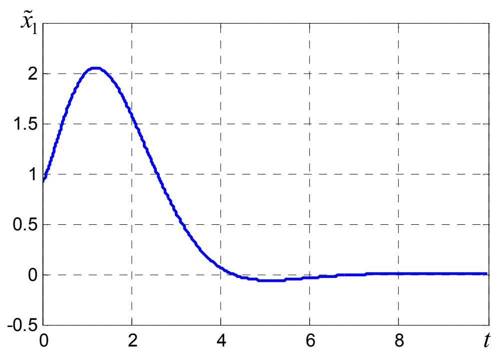

Fig.1 – State variable of the optimal system

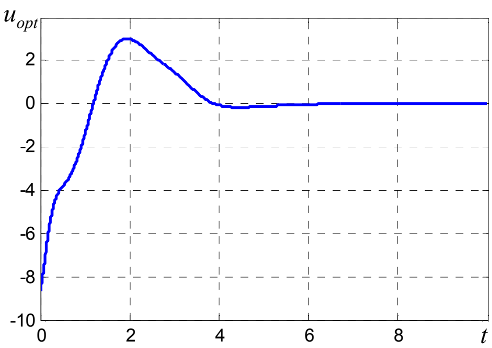

Fig.2 – Optimal control

The submitted plot are received by simulation in MATLAB of the optimal control system (33), (38) (40), (41) with initial conditions $\tilde{x}_0 = [0,9 \quad 0,5 \quad 0]^T$.

4 Conclusion

If the equations of a nonlinear plant are convertible to Jordan controlled form, it is possible to find the stabilizing control or the control which is optimal in the sense of a minimum of uncertain nonlinear quadratic criteria. The optimal control is created in two steps. First, the linearization control is designed. Further, the optimal control is determined by the solution of the known Riccati equation. Proposed procedure of the optimal control system design is completely analytical. Desirable character of the optimal system transients can be found by change of the nonlinear quadratic criteria coefficients. The example of the optimal control system design is presented.

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