# The estimation of the m parameter of the Nakagami distribution

Li-Fei Huang
Ming Chuan University
Dept. of Applied Stat. and Info. Science
5 Teh-Ming Rd., Gwei-Shan Taoyuan City
Taiwan
lhuang@mail.mcu.edu.tw

Jen-Jen Lin
Ming Chuan University
Dept. of Applied Stat. and Info. Science
5 Teh-Ming Rd., Gwei-Shan Taoyuan City
Taiwan
jjlin@mail.mcu.edu.tw

*Abstract:* This paper introduces the Nakagami-m distribution which is usually used to simulate the ultrasound image. The gamma distribution is used to derive the moment estimator and the maximum likelihood estimator because the Nakagami distribution has no moment generation function and too complicate likelihood function. The moment estimator of m is normally distributed with a smaller bias and a larger standard deviation. The second order maximum likelihood estimator of m is also normally distributed with a larger bias and a smaller standard deviation. The confidence interval for the ratio of medians from two independent distributions of Nakagami-m estimators is constructed. The moment estimator provides a quick understanding about the m parameter, while the second order maximum likelihood estimator provides a full understanding about the m parameter.

*Key–Words:* The Nakagami distribution, The moment estimator, The maximum likelihood estimator, The distribution of the Nakagami-m, The ratio of medians.

## 1 Introduction

The Nakagami distribution is usually used to simulate the ultrasound image. Smolikova et al. say that analysis of backscatter in the ultrasound echo envelope, in conjunction with ultrasound B-scans, can provide important information for tissue characterization and pathology diagnosis[6]. Pavlovic et al. present the joint probability density function (PDF) and PDF of maximum of ratios mu(1) = R-1/r(1) and mu(2) = R-2/r(2) for the cases where R-1, R-2, r(1), and r(2) are Rayleigh, Rician, Nakagami-m., and Weibull distributed random variables[4]. Agrawal and Karmeshu propose a new composite probability distribution, i. e. Nakagami-generalized inverse Gaussian distribution (NGIGD) with four parameters[1]. Chen applies order statistics to analyze the performance of ordered selection combining schemes with different modulation receptions operating in Nakagami-m fading environments and all the results are validated by comparing the special case Rayleigh distribution with the fading figure m = 1 in the Nakagami-m distribution[2]. Peppas presents a closed-form expression for the moments generating function of the half-harmonic mean of two independent, not necessarily identically distributed gamma random variables with arbitrary parameters[5].

With the shape parameter $m$ and the spread parameter $\Omega$, the probability density function of the Nakagami distribution is as follows.

$$f(n) = \frac{2}{\Gamma(m)}(\frac{m}{\Omega})^m n^{2m-1} e^{-\frac{m}{\Omega}n^2}$$

Huang and Johnson [3] provide theorems to construct a confidence interval for ratio of percentiles from two independent distributions.

## 2 The estimators of Nakagami-m

### 2.1 The moment estimator

If $N \sim$ Nakagami$(m, \Omega)$, let $G = N^2$. Then $n = g^{1/2}$ and $G = N^2 \sim$ gamma$(m, \frac{\Omega}{m})$.

$$|J| = |\frac{dn}{dg}| = |\frac{1}{2}g^{-\frac{1}{2}}|$$

$$f(n) = \frac{2}{\Gamma(m)}(\frac{m}{\Omega})^m n^{2m-1} e^{-\frac{m}{\Omega}n^2}$$

$$\Rightarrow f(g) = \frac{2}{\Gamma(m)}(\frac{m}{\Omega})^m g^{\frac{2m-1}{2}} e^{-\frac{m}{\Omega}g} \cdot \frac{1}{2}g^{-\frac{1}{2}}$$

$$= \frac{1}{\Gamma(m)}(\frac{m}{\Omega})^m g^{m-1} e^{-\frac{m}{\Omega}g}$$

With the shape parameter $\alpha$ and the scale parameter $\beta$, the probability density function of the gamma distribution is as follows.

$$f(g) = \frac{1}{\Gamma(\alpha)\beta^\alpha} g^{\alpha-1} e^{-\frac{g}{\beta}}$$

Table 1: The moment estimator of $m$

| $m$ | $\Omega$ | $\hat{m}$ |
|---|---|---|
| 1 | 1 | 1.003686 |
| 1 | 291848 | 1.003686 |
| 0.75 | 1 | 0.7531207 |
| 0.75 | 291848 | 0.7531207 |
| 0.5 | 1 | 0.5006322 |
| 0.5 | 291848 | 0.5006322 |
| 0.25 | 1 | 0.2493589 |
| 0.25 | 291848 | 0.2493589 |

The moment generating function:

$$M_G(t) = E(e^{tg})$$

$$= \int_0^\infty \frac{1}{\Gamma(\alpha)\beta^\alpha} g^{\alpha-1} e^{-(\frac{1}{\beta}-t)g} dg$$

$$= \int_0^\infty \frac{1}{\Gamma(\alpha)\beta^\alpha} g^{\alpha-1} e^{-(\frac{1-\beta t}{\beta})g} dg$$

$$= (\frac{1}{1-\beta t})^\alpha$$

$$E(G) = M'_G(0) = -\alpha(1-\beta t)^{-\alpha-1}(-\beta)|_{t=0} = \alpha\beta$$

$$E(G^2) = M''_G(0) = \alpha\beta^2(\alpha+1)$$

$$= \alpha\beta(-\alpha-1)(1-\beta t)^{-\alpha-2}(-\beta)|_{t=0}$$

$$Var(G) = E(G^2) - E(G)^2 = \alpha\beta^2(\alpha+1-\alpha) = \alpha\beta^2$$

$$\alpha = \frac{E(G)^2}{Var(G)}, \beta = \frac{Var(G)}{E(G)}$$

It can be shown that

$$m = \frac{E(N^2)^2}{Var(N^2)}.$$

Since the moment generation function of the Nakagami distribution does not exist, it's necessary to apply the gamma distribution in the process of finding the moment estimator of $m$.

If a random sample of $n_1$, $n_2$, $\ldots$, $n_{1000}$ is selected spread parameter $\Omega$, the moment estimators of shape parameters are listed in Table 1. The value of $\Omega$ won't affect the estimation of $m$.

## 2.2 The maximum likelihood estimator

Let $n$ be the sample size. The likelihood function is

$$L(\alpha,\beta) = \prod_{i=1}^n f(g_i|\alpha,\beta)$$

$$= \prod_{i=1}^n \frac{1}{\Gamma(\alpha)\beta^\alpha} g_i^{\alpha-1} e^{-\frac{g_i}{\beta}}$$

$$= \{\frac{1}{\Gamma(\alpha)\beta^\alpha}\}^n \{\prod_{i=1}^n g_i\}^{\alpha-1} e^{-\frac{\sum_{i=1}^n g_i}{\beta}}.$$

The log likelihood function is

$$\ln L(\alpha,\beta) = -n\ln\{\Gamma(\alpha)\beta^\alpha\}$$

$$+(\alpha-1)\sum_{i=1}^n \ln g_i - \frac{1}{\beta}\sum_{i=1}^n g_i$$

$$\frac{\partial \ln L(\alpha,\beta)}{\partial\beta} = \frac{-n\alpha}{\beta} + \frac{\sum_{i=1}^n g_i}{\beta^2} = 0$$

$$\Rightarrow \hat{\Omega} = \frac{\sum_{i=1}^n g_i}{n},$$

which is the same with its moment estimator.

$$\frac{\partial \ln L(\alpha,\beta)}{\partial\alpha} = -n \cdot \frac{\Gamma'(\alpha)}{\Gamma(\alpha)} - n\ln\beta + \sum_{i=1}^n \ln g_i$$

$$= -n \cdot \frac{\Gamma'(\alpha)}{\Gamma(\alpha)} - n\ln\{\frac{\sum_{i=1}^n g_i}{n\alpha}\} + \sum_{i=1}^n \ln g_i = 0$$

$$\Rightarrow -\frac{\Gamma'(\alpha)}{\Gamma(\alpha)} + \ln\alpha = \ln\{\frac{\sum_{i=1}^n g_i}{n}\} - \frac{\sum_{i=1}^n \ln g_i}{n}$$

The digamma function is defined to be

$$\psi(\alpha) = \frac{d}{d\alpha}\ln\Gamma(\alpha) = \frac{\Gamma'(\alpha)}{\Gamma(\alpha)}.$$

Now we only need to deal with the digamma function. The likelihood function of the Nakagami distribution is too complicate to find the maximum likelihood estimator of $m$. Let

$$-\psi(\alpha) + \ln\alpha = \triangle = \ln\{\frac{\sum_{i=1}^n g_i}{n}\} - \frac{\sum_{i=1}^n \ln g_i}{n}.$$

Abramowitz and Stegun find that

$$-\psi(\alpha) + \ln\alpha = \triangle \approx \frac{1}{2\alpha} + \frac{1}{12\alpha^2} - \frac{1}{120\alpha^4} + \cdots.$$

One term is used to obtain the first order maximum likelihood estimator.

$$\triangle = \frac{1}{2\hat{\alpha}_1} \Rightarrow \hat{m}_1 = \hat{\alpha}_1 = \frac{1}{2\triangle}$$

Table 2: The 1st and 2nd order maximum likelihood estimator of $m$

| $m$ | $\Omega$ | $\hat{m}_1$ | $\hat{m}_2$ |
|---|---|---|---|
| 1 | 1 | 0.8687865 | 1.001858 |
| 1 | 291848 | 0.8687865 | 1.001858 |
| 0.75 | 1 | 0.628175 | 0.7499995 |
| 0.75 | 291848 | 0.628175 | 0.7499995 |
| 0.5 | 1 | 0.3945828 | 0.4998179 |
| 0.5 | 291848 | 0.3945828 | 0.4998179 |
| 0.25 | 1 | 0.1763376 | 0.2499231 |
| 0.25 | 291848 | 0.1763376 | 0.2499231 |

Two terms are used to obtain the second order maximum likelihood estimator.

$$\triangle = \frac{1}{2\hat{\alpha}_2} + \frac{1}{12\hat{\alpha}_2^2} \Rightarrow 12\Delta\hat{\alpha}_2^2 = 6\hat{\alpha}_2 + 1 \Rightarrow$$

$$12\Delta\hat{\alpha}_2^2 - 6\hat{\alpha}_2 - 1 = 0 \Rightarrow \hat{m}_2 = \hat{\alpha}_2 = \frac{3 + \sqrt{9 + 12\triangle}}{12\triangle}$$

If a random sample of $n_1$, $n_2$, $\ldots$, $n_{1000}$ is selected from a Nakagami distribution with the shape parameter $m$ and the spread parameter $\Omega$, the maximum likelihood estimators of shape parameters are listed in Table 2. Where $\hat{m}_2$ is adjusted by its bias. The value of $\Omega$ won't affect the estimation of $m$.

# 3 The distribution of the Nakagami-m estimator

In order to compare the moment estimator and the maximum likelihood estimator, their distributions need to be found. The Kolmogorov-Smirnov test is applied to find the distribution of the Nakagami-m estimator.

## 3.1 The moment estimator

The moment estimator is normally distributed with a smaller bias and a larger standard deviation.

## 3.2 The maximum likelihood estimator

The first order maximum likelihood estimator is not discussed because it's too far away from the theoretical value of $m$. The second order maximum likelihood estimator is normally distributed with a larger bias and a smaller standard deviation.

Table 3: The bias and the standard deviation of the moment estimator of $m$

| $m$ | $\Omega$ | bias of $\hat{m}$ | stdev. of $\hat{m}$ |
|---|---|---|---|
| 1 | 1 | 0 | 0.06 |
| 1 | 291848 | 0 | 0.06 |
| 0.75 | 1 | 0 | 0.05 |
| 0.75 | 291848 | 0 | 0.05 |
| 0.5 | 1 | 0.002 | 0.04 |
| 0.5 | 291848 | 0.002 | 0.04 |
| 0.25 | 1 | 0.003 | 0.025 |
| 0.25 | 291848 | 0.003 | 0.025 |

Table 4: The bias and the standard deviation of the second order maximum likelihood estimator of $m$

| $m$ | $\Omega$ | bias of $\hat{m}_2$ | stdev. of $\hat{m}_2$ |
|---|---|---|---|
| 1 | 1 | 0.01 | 0.04 |
| 1 | 291848 | 0.01 | 0.04 |
| 0.75 | 1 | 0.015 | 0.03 |
| 0.75 | 291848 | 0.015 | 0.03 |
| 0.5 | 1 | 0.021 | 0.02 |
| 0.5 | 291848 | 0.021 | 0.02 |
| 0.25 | 1 | 0.031 | 0.0075 |
| 0.25 | 291848 | 0.031 | 0.0075 |

Table 5: The Cramer-Rao lower bound for the moment estimator of $m$

| $m$ | bias | $I(m)$ | $\sqrt{CRLB}$ | stdev. |
|---|---|---|---|---|
| 1 | 0 | 333177.9 | 0.0017 | 0.06 |
| 0.75 | 0 | 785200.9 | 0.0011 | 0.05 |
| 0.5 | 0.002 | 1227203.0 | 0.00089 | 0.04 |
| 0.25 | 0.003 | 1359551.0 | 0.00085 | 0.025 |

Table 6: The Cramer-Rao lower bound for the second order maximum likelihood estimator of $m$

| $m$ | bias | $I(m)$ | $\sqrt{CRLB}$ | stdev. |
|---|---|---|---|---|
| 1 | 0.01 | 333177.9 | 0.0017 | 0.04 |
| 0.75 | 0.015 | 785200.9 | 0.0011 | 0.03 |
| 0.5 | 0.021 | 1227203.0 | 0.00088 | 0.02 |
| 0.25 | 0.031 | 1359551.0 | 0.00082 | 0.0075 |

## 3.3 The Cramer-Rao lower bound

Any estimator $\hat{\alpha}$ whose bias is given by a function $b(\alpha)$ satisfies the Cramer-Rao lower bound

$$Var(\hat{\alpha}) \geq \frac{[1+b^{'}(\alpha)]^2}{I(\alpha)},$$

where

$$I(\alpha) = E\{[\frac{\partial \ln L(\alpha,\beta)}{\partial \alpha}]^2\}$$

$$= E\{[-n \cdot \frac{\Gamma^{'}(\alpha)}{\Gamma(\alpha)} - n \ln\{\frac{\sum_{i=1}^{n} g_i}{n\alpha}\} + \sum_{i=1}^{n} \ln g_i]^2\}$$

$$= E\{[-n \cdot \frac{\Gamma^{'}(\alpha)}{\Gamma(\alpha)} + n \ln \alpha - n\triangle]^2\}$$

The square root of Cramer-Rao lower bounds for the moment estimators are summarized in the Table 5.

The square root of Cramer-Rao lower bounds for the second order maximum likelihood estimators are summarized in the Table 6.

## 3.4 The median of the Nakagami-m estimators

In order to compare population medians, the confidence intervals for the ratio of two medians from independent distributions of Nakagami-m estimators are constructed by way of two rewritten theorems based on nonparametric methods in Huang and Johnson [3]. $m$ and $n$ in the theorems are just sample sizes.

**Theorem 1** *Let $X_1$, $X_2$, ..., $X_m$ be a random sample from $F_1(\cdot)$; let $Y_1$, $Y_2$, ..., $Y_n$ from $F_2(\cdot)$; and let samples be independent. If the population density function $F_i^{'}(\cdot)$, is positive and continuous in a neighborhood of the median $m_i$, for i=1,2, $\lim_{m,n\to\infty} m(m+n) = \lambda$, $(0 < \lambda < 1)$, and $m_2 \neq 0$, then the 100(1-$\alpha$)% confidence interval of $\theta = m_1/m_2$ is*

$$\frac{\hat{m}_1}{\hat{m}_2} \pm Z_{\alpha/2}\sqrt{\frac{0.25}{m\hat{m}_2^2[\hat{F}_1^{'}(\hat{m}_1)]^2} + \frac{0.25\hat{m}_1^2}{n\hat{m}_2^4[\hat{F}_2^{'}(\hat{m}_2)]^2}}$$

**Theorem 2** *Let $X_1$, $X_2$, ..., $X_m$ be a random sample from $F_1(\cdot)$ which has a positive continuous derivative $F_1^{'}(\cdot)$ in a neighborhood of $m_1$, let $Y_1$, $Y_2$, ..., $Y_n$ be a random sample from $F_2(\cdot)$ which has a positive continuous derivative $F_2^{'}(\cdot)$ in a neighborhood of $m_2$, and let two samples be independent. Let the ratio of medians $\theta = m_1/m_2$ be unknown but finite. Under the condition that $\lim_{m,n\to\infty} m(m+n) = \lambda$, $(0 < \lambda < 1)$, $\theta$ values of the intersections of $\pm Z_{\alpha/2}$ and*

$$Z^{M,NP}(\theta) = \frac{\hat{m}_1 - \theta\hat{m}_2}{\sqrt{\frac{0.25}{m[\hat{F}_1^{'}(\hat{m}_1)]^2} + \frac{0.25\theta^2}{n[\hat{F}_2^{'}(\hat{m}_2)]^2}}}$$

*form the 100(1-$\alpha$)% confidence interval.*

Medians are estimated by order statistics, and $\hat{F}_i^{'}(\hat{m}_i)$ is estimated by the kernel estimator for $i$=1,2. One thousand replications of the random sample $X_{i,1}$, $X_{i,2}$, ..., $X_{i,1000}$ are selected independently from a Nakagami distribution with the shape parameter $m$ and the spread parameter 1. One thousand replications of the random sample $Y_{i,1}$, $Y_{i,2}$, ..., $Y_{i,1000}$ are selected independently from a Nakagami distribution with the shape parameter 1 and the spread parameter 1. The true ratio of medians from two populations should be $m$. The simulated 95% confidence intervals are listed in the Table 7. Where the moment estimator and the second order maximum likelihood estimator are adjusted by their bias.

## 3.5 The advantage and disadvantage of two kinds of estimators

The moment estimator has two advantages. At first, it's much easier to find the moment estimator than the maximum likelihood estimator. Secondly, the bias of the moment estimator is smaller than the bias of the maximum likelihood estimator.

The maximum likelihood estimator also has two advantages. At first, the standard deviation of the maximum likelihood estimator is closer to the Cramer-Rao lower bound than the standard deviation

Table 7: The 95% confidence interval of $m$

| $m$ | Thm. | Estimator | 95% CI |
|---|---|---|---|
| 1 | 1 | MME | (0.9967,1.0109) |
| 1 | 1 | MLE2 | (0.9993,1.0078) |
| 1 | 2 | MME | (0.9970,1.0110) |
| 1 | 2 | MLE2 | (0.9990,1.0080) |
| 0.75 | 1 | MME | (0.7487,0.7599) |
| 0.75 | 1 | MLE2 | (0.7479,0.7542) |
| 0.75 | 2 | MME | (0.7490,0.7600) |
| 0.75 | 2 | MLE2 | (0.7480,0.7540) |
| 0.5 | 1 | MME | (0.4980,0.5060) |
| 0.5 | 1 | MLE2 | (0.4984,0.5025) |
| 0.5 | 2 | MME | (0.4980,0.5060) |
| 0.5 | 2 | MLE2 | (0.4980,0.5030) |
| 0.25 | 1 | MME | (0.2479,0.2523) |
| 0.25 | 1 | MLE2 | (0.2492,0.2512) |
| 0.25 | 2 | MME | (0.2480,0.2520) |
| 0.25 | 2 | MLE2 | (0.2490,0.2510) |

of the moment estimator. Secondly, the confidence interval constructed by the maximum likelihood estimator is shorter than the confidence interval constructed by the moment estimator under all circumstances.

# 4 Conclusion

The gamma distribution can be used to derive and estimate the shape parameter of the Nakagami distribution. The confidence interval for the ratio of medians from two independent distributions of the Nakagami-m estimators can be constructed. Because the moment estimator of $m$ is easier to be calculated and has a smaller bias, a quick understanding about $m$ can be obtained by finding its moment estimator. Since the maximum likelihood estimator of $m$ has a smaller standard deviation, a full understanding about $m$ can be obtained by finding its maximum likelihood estimator. R programs of the Nakagami variable simulation and parameters estimation and the confidence interval construction are in the appendix.

*References:*


[1] R. Agrawal and Karmeshu, Ultrasonic backscattering in tissue: characterization through Nakagami-generalized inverse Gaussian distribution, *Computers in Biology and Medicine* 37(2), 2007, pp. 166–172.

[2] JLZ. Chen, Performance analysis for ordered selection combining schemes in Nakagami-m environments, *Journal of the Franklin Institute-Engineering and Applied Mathematics* 342(6), 2005, pp. 638–656.

[3] LF. Huang and RA. Johnson, Confidence regions for the ratio of percentiles, *Statistics and Probability Letters* 76(4), 2006, pp. 384–492.

[4] DC. Pavlovic, NM. Sekulovic, GV. Milovanovic, AS. Panajotovic, MC. Stefanovic and ZJ. Popovic, Statistics for Ratios of Rayleigh, Rician, Nakagami-m, and Weibull Distributed Random Variables, *Mathematical Problems in Engineering* 2013, DOI: 10. 1155/2013/252804 .

[5] K. Peppas, Moments generating function of the harmonic mean of two non-identical gamma random variables and its applications in wireless communications, *Journal of the Franklin Institute-Engineering and Applied Mathematics* 349(3), 2012, pp. 845–860.

[6] R. Smolikova, MP. Wachowiak and JM. Zurada, An information-theoretic approach to estimating ultrasound backscatter characteristics, *Computers in Biology and Medicine* 34(4), 2004, pp. 355–370.


**Appendix**

A.1 R program of the Nakagami variable simulation and parameters estimation

```
DistTest=function(Simulated,
NM,BigOmega,WriteTo,
MMEbias,MMEstdev,
MLE2bias,MLE2stdev)
{
 data=read.table(Simulated)
 data.mat=as.matrix(data)
 raw=matrix(0,nrow(data),
 ncol(data))
 Be=NM/BigOmega
 for (i in 1:nrow(data))
 {
  for (j in 1:ncol(data))
  {
   raw[i,j]=qgamma(data.mat[i,
   j], NM, Be)
  }
 }
 nakagami=matrix(0,nrow(data),
 ncol(data))
 for (i in 1:nrow(data))
 {
  for (j in 1:ncol(data))
```

```
   {
    nakagami[i,j]=sqrt(raw[i,j])
   }
  }
 mEsti=matrix(0,nrow(data),7)
 TmEsti=matrix(0,2,nrow(data))
 for (i in 1:nrow(data))
 {
  mEsti[i,1]=mean(raw[i,])^2/
  var(raw[i,])-MMEbias
  mEsti[i,5]=1/(2*(log(mean(raw[
  i,]))-mean(log(raw[i,]))))
  mEsti[i,6]=(3+sqrt(9+12*(log(
  mean(raw[i,]))-mean(log(raw[i,
  ]))))
)/(12*(log(mean(raw[i,]))-mean(
log(raw[i,]))))-MLE2bias
  mEsti[i,7]=-ncol(data)*digamma(
  NM)/gamma(NM)-ncol(data)*log(
  1/Be)+sum(log(raw[i,]))
 }
 for (i in 1:nrow(data))
 {
  TmEsti[1,i]=mEsti[i,1]
  TmEsti[2,i]=mEsti[i,6]
 }
 write.table(TmEsti,WriteTo,
 col.names=FALSE,
 row.names=FALSE, sep=" ")
 out4=mean(mEsti[,1])
 out7=sqrt(var(mEsti[,1]))
 ksRes=ks.test(mEsti[,1],"
 pnorm",NM,MMEstdev)
 out8=ksRes$p.value
 out13=mean(mEsti[,5])
 out14=mean(mEsti[,6])
 out15=sqrt(var(mEsti[,5]))
 out16=sqrt(var(mEsti[,6]))
 ksRes2=ks.test(mEsti[,6],
 "pnorm",NM,MLE2stdev)
 out18=ksRes2$p.value
 out19=mean((mEsti[,7])^2)
 list(meanMME=out4,
sdMME=out7,NormalP=out8
 ,meanMLE1=out13,meanMLE2=
out14,sdMLE1=out15,sdMLE2=out16
 ,NormalP2=out18,FisherI=out19
)
}
```

### A.2 R program of the confidence interval construction

```
RoMCI=function(trueRatio,numer,
denom, WriteTo1, WriteTo2
###, Thm1, Thm2
)
{
mal=read.table(numer,header=
FALSE, sep=" ")
ben=read.table(denom,header=
FALSE, sep=" ")
malsize=ncol(mal)
bensize=ncol(ben)
npairs=nrow(mal)
malmat=as.matrix(mal)
benmat=as.matrix(ben)

malker=matrix(0,npairs,1+
malsize)
benker=matrix(0,npairs,1+
bensize)
Thm32=matrix(0,npairs,4)
Thm42=matrix(0,npairs,4)
ztheta=rep(0,11)

for (i in 1:npairs)
{
malker[i,1+malsize]=median(
malmat[i,])
benker[i,1+bensize]=median(
benmat[i,])
sortmal=sort(malmat[i,])
sortben=sort(benmat[i,])
malup=sortmal[ceiling(malsize/
2)+1:malsize]
mallow=sortmal[1:floor(malsize/
2)]
benup=sortben[ceiling(bensize/
2)+1:bensize]
benlow=sortben[1:floor(bensize/
2)]
a1=c(sqrt(apply(mal[i,],1,var)),
(median(malup[1:floor(malsize/
2)])-median(mallow))/1.34)
A1=min(a1)
a2=c(sqrt(apply(ben[i,],1,var)),
(median(benup[1:floor(bensize/
2)])-median(benlow))/1.34)
A2=min(a2)
h1=0.9*A1*malsize^(-0.2)
h2=0.9*A2*bensize^(-0.2)
for (j in 1:malsize)
{
malker[i,j]=exp(-((malker[i,
1+malsize]-malmat[i,j])/h1)^2/
2)/(sqrt(2*pi)*malsize*h1)
}
for (k in 1:bensize)
{
benker[i,k]=exp(-((benker[i,
```

```
1+bensize]-benmat[i,k])/h2)^2/
2)/(sqrt(2*pi)*bensize*h2)
}
Thm32[i,2]=malker[i,1+malsize]/
benker[i,1+bensize]
Thm32[i,1]=Thm32[i,2]-1.96*
sqrt(0.25/(malsize*benker[i,
1+bensize]^2
*sum(malker[i,1:malsize])^2)+
0.25*malker[i,1+malsize]^2/
(bensize*benker[i,1+bensize]^4
*sum(benker[i,1:bensize])^2))
Thm32[i,3]=Thm32[i,2]+(Thm32[
i,2]-Thm32[i,1])
Thm32[i,4]=Thm32[i,1]<trueRatio &
Thm32[i,3]>trueRatio

Thm42[i,2]=Thm32[i,2]

for (l in 1:11)
{
theta=0.1*(l-1)
ztheta[l]=(malker[i,1+malsize]-
theta*benker[i,1+bensize])/
sqrt(0.25/(malsize*sum(malker[i,
1:malsize])^2)+0.25
*theta^2/(bensize*sum(benker[i,
1:bensize])^2))
}
ltheta=ztheta[ztheta>1.96]
utheta=ztheta[ztheta>-1.96]
lback1=length(ltheta)-1
uback1=length(utheta)-1

for (l in 1:11)
{
theta=0.1*lback1+0.01*(l-1)
ztheta[l]=(malker[i,1+malsize]-
theta*benker[i,1+bensize])/
sqrt(0.25/(malsize*sum(malker[
i,1:malsize])^2)+0.25
*theta^2/(bensize*sum(benker[i,
1:bensize])^2))
}
ltheta=ztheta[ztheta>1.96]
lback2=length(ltheta)-1

for (l in 1:11)
{
theta=0.1*lback1+0.01*lback2+
0.001*(l-1)
ztheta[l]=(malker[i,1+malsize]-
theta*benker[i,1+bensize])/
sqrt(0.25/(malsize*sum(malker[
i,1:malsize])^2)+0.25
*theta^2/(bensize*sum(benker[
i,1:bensize])^2))
}
ltheta=ztheta[ztheta>1.96]
judge= ztheta[length(ltheta)]-
1.96<1.96-ztheta[length(ltheta
)+1]
lback3=length(ltheta)-judge

Thm42[i,1]=0.1*lback1+0.01*
lback2+0.001*lback3

for (l in 1:11)
{
theta=0.1*uback1+0.01*(l-1)
ztheta[l]=(malker[i,1+malsize]-
theta*benker[i,1+bensize])/
sqrt(0.25/(malsize*sum(malker[
i,1:malsize])^2)+0.25
*theta^2/(bensize*sum(benker[
i,1:bensize])^2))
}
utheta=ztheta[ztheta>-1.96]
uback2=length(utheta)-1

for (l in 1:11)
{
theta=0.1*uback1+0.01*uback2+
0.001*(l-1)
ztheta[l]=(malker[i,1+malsize]-
theta*benker[i,1+bensize])/
sqrt(0.25/(malsize*sum(malker[
i,1:malsize])^2)+0.25
*theta^2/(bensize*sum(benker[
i,1:bensize])^2))
}
utheta=ztheta[ztheta>-1.96]
judge= ztheta[length(utheta)]+
1.96< -1.96-ztheta[length(utheta
)+1]
uback3=length(utheta)-judge

Thm42[i,3]=0.1*uback1+0.01*
uback2+0.001*uback3

Thm42[i,4]=Thm42[i,1]<trueRatio &
Thm42[i,3]>trueRatio
}
write.table(Thm32, WriteTo1,
col.names=FALSE, row.names=FALSE,
sep=" ")
write.table(Thm42, WriteTo2,
col.names=FALSE, row.names=FALSE,
sep=" ")
}
```