Some Solutions to a System of Equations modelled for Viral Dynamics subsequent to a Liver Transplantation in Patients with Chronic Hepatitis B and D

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Abstract: Patients who lost livers through cirrhosiss resulting from a Hepatitis virus, are at risk of reinfection after transplantation. Blood processes dynamics are good indicators of this occurring again. In this contribution, we study a mechanistic model, on these blood processes, consisting of a system of ordinary differential equations. We first obtain simple solution through variation of parameters, by exploiting an error at the core of Euler's ansatz for solving linear ordinary differential equations. Once that is done, we then fully explore the model fully through symmetry analysis. The traditional regular symmetries usually lead to expressions that are impossible to integrate, subsequently forcing the analyst to consider special cases that may not even be practical. Here we modify the symmetries to avert this.

Key–Words: Infectious diseases, Ordinary differential equations, Symmetry analysis, Variation of parameters, Viral dynamics

1 Introduction

There are five known hepatitis viruses. that is, A, B, C, D, and E, or HAV, HBV, HCV, HDV and HEV. The HDV can only propagate in the presence of HBV, hence the need to study both. The model we address here, that we want to solve, is borrowed from Filmann and Herrmann[1], and has the form

$$\dot{V} = pI - cV, \tag{1}$$
$$\dot{I} = \beta TV - \delta I, \tag{2}$$
$$\dot{T} = \lambda - \beta TV - dT \tag{3}$$

In this model there are three dependent variables, namely T, I and V. The variable T represents the size of the uninfected cell population. The variable I denotes the infected cells, while V is the free virus particles in serum. It is assumed the uninfected cells are produced at a rate λ and die at the rate d, by V . Uninfected cells T are assumed to be produced at a constant rate and to die at a rate d. The free virus particles V are produced at a rate p proportional to I and are removed from the system at a rate c. Target cells T are infected at a rate β proportional to TV . Infected cells I are killed by the immune system at a rate δ.

We intend solving the model using Lie's symmetry group theoretical methods, a technique first introduced by Marius Sophus Lie (1842 – 1899). That is, a slightly modified version thereof. The pure Lie approach tend to run into difficulties. In most studies, the symmetry groups never materials, thus rendering the whole exercise futile. Where they exist other difficulties are encountered. For example, the analyses lead to integrals that cannot be evaluated. Some practitioners tend to avoid these situations by modifying the models parameters. Unfortunately such acts tend to have adverse effects on applications. For a further read on the theory, and its applications to other fields, one is referred to Kallianpur, and Karandikar [2], Kwok [2], Hui [3], Longstaff [4], Platen [5], Naicker, Andriopoulos, and Leach [6], Pooe, Mahomed, and Soh [7], Sinkala, Leach, and OHara [8], Gazizov, and Ibragimov [9]. We believe we may have found a remedy. This we discuss in the next section.

Section 2 is divided into three subsections. In the first subsection, Subsection 2.1, we briefly outline the basic principles of Lie's theory, first introduced through his now famous 1881 paper [10]. This we do to ease comparison. Next, in Subsection 2.2, we show where in Lie's theory our modifications fit. We conclude the section by providing a simple formula in Subsection 2.3, for generating the proposed symmetries.

Section 3 is on the actual application of the ideas discussed in Subsection 2.2 to the system of equations (1), (2) and (3). It starts off in the traditional Lie fashion, then the formula discussed in Section 2.3 is gradually introduced.

The subject discussed in Section 4 is included for a number of reasons. It is used as a tool for generating parallel empirical data, through which the symmetry method can be confirmed for validity, as opposed to using numerical techniques. The latter proceeds through steps, and as such, tends to step over important features like singularities, hiding them from possible detection. It can also be used to suggest alternative symmetries by confirming solvability, when the basic symmetry approach fails, as alluded to in Section 2, which is not the case here.

2 Theoretical Basis for Modified Symmetries

Smart symmetries, or modified one-parameter local point symmetries in this case, or simply modified symmetries for short, is a new concept that we are introducing, and want others to try. It is for this reason that we see a need for more depth and details. We first present the traditional approach.

2.1 Traditional symmetries

By Traditional symmetries here we are referring to local one-parameter point transformations, and not all symmetries in general. A broader discussion would take a lot of space. In here, we dwell on symmetries that apply to second order ordinary differential equations.

To begin, we first define a group.

Definition 1 *A group G is a set of elements with a law of composition ϕ between elements satisfying the following axioms:*

(i) Closure. For $\{G_1, G_2\} \subset G$, we have $\phi(G_1, G_2) \in G$.

(ii) Associativity. For $\{G_1, G_2, G_2\} \subset G$, we have $\phi(G_1, \phi(G_2, G_3)) = \phi(\phi(G_1, G_2), G_3) \in G$.

(iii) Identity. There exists $G_0 \in G$, such that $\phi(G_0, G_i) = \phi(G_i, G_0) = G_i$, for every element G_i in G. The element G_0 is called the identity element of G.

(iv) Inverse. There exists $G_i^{-1} \in G$ for every $G_i \in G$, such that $\phi(G_i^{-1}, G_i) = \phi(G_i, G_i^{-1}) = G_0 \in G$. The element G_i^{-1} is called the inverse of G_i.

That done, we next turn to group of transformations.

Definition 2 *Let*

$$\bar{\mathbf{x}} = \psi\left(\mathbf{x}; \epsilon\right) \tag{4}$$

be a family of invertible transformations, of points $\mathbf{x} = (x^1, \cdots, x^N) \in D \subset \mathrm{I\!R}^N$ into points $\bar{\mathbf{x}} = (\bar{x}^1, \cdots, \bar{x}^N) \in R \subset \mathrm{I\!R}^{\mathbf{N}}$, with the parameter $\epsilon \in S \subset \mathrm{I\!R}$. These are called one-parameter group of point transformations if the following hold.

(i) For each $\epsilon \in S$, we have the transformations being one-to-one and onto D, meaning D is not different from R, as x^N is not different from $\bar{x}^N$.

(ii) The set S is a group, say G, with $\phi(\epsilon, \delta)$ defining the composition law.

(iii) The case $\bar{\mathbf{x}} = \mathbf{x}$ corresponds to $\epsilon = \epsilon_0$: The identity element of G. That is,

$$\bar{\mathbf{x}}|_{\epsilon=\epsilon_0} = \mathbf{x}. \tag{5}$$

or,

$$\psi\left(\mathbf{x}; \epsilon\right)\Big|_{\epsilon=\epsilon_0} = \mathbf{x}. \tag{6}$$

(iv) If $\bar{\mathbf{x}} = \psi\left(\mathbf{x}; \epsilon\right)$ and $\bar{\bar{\mathbf{x}}} = \psi\left(\bar{\mathbf{x}}; \delta\right)$, then

$$\bar{\bar{\mathbf{x}}} = \psi\left(\mathbf{x}; \phi(\epsilon, \delta)\right).$$

Theorem 1 Lie's First Fundamental Theorem: *There exists a parametrization $\tau(\epsilon)$ such that the Lie group of transformations is equivalent to the solution of an initial value problem for a system of first-order ODEs given by*

$$\frac{d\bar{x}}{d\tau} = \xi(\bar{x}), \tag{7}$$

with

$$\frac{d\bar{x}}{d\tau}\Big|_{\tau=0} = x. \tag{8}$$

2.1.1 Local one-parameter point transformation groups

The transformation can be expanded using the Taylor-Maclaurin series expansion with respect to the parameter. That is,

$$\bar{\mathbf{x}} = \mathbf{x} + \epsilon \left(\left. \frac{\partial \mathbf{G}}{\partial \epsilon} \right|_{\epsilon=0} \right) + \frac{\epsilon^2}{2} \left(\left. \frac{\partial^2 \mathbf{G}}{\partial \epsilon^2} \right|_{\epsilon=0} \right) + \cdots = \mathbf{x} + \epsilon \left(\left. \frac{\partial \mathbf{G}}{\partial \epsilon} \right|_{\epsilon=0} \right) + O\left(\epsilon^2\right). \quad (9)$$

Letting

$$\xi(\mathbf{x}) = \left. \frac{\partial \mathbf{G}}{\partial \epsilon} \right|_{\epsilon=0}, \quad (10)$$

reduces the expansion to

$$\bar{\mathbf{x}} = \mathbf{x} + \epsilon \xi(\mathbf{x}) + O\left(\epsilon^2\right). \quad (11)$$

Definition 3 *The expression*

$$\bar{\mathbf{x}} = \mathbf{x} + \epsilon \xi(\mathbf{x}), \quad (12)$$

is called a local one-parameter point transformation.

The set G is a group since the following properties hold under binary operation $+$:

1. **Closure**. If $\bar{\mathbf{x}}_{\epsilon_1}, \bar{\mathbf{x}}_{\epsilon_2} \in G$ and $\epsilon_1, \epsilon_2 \in \mathbb{R}$, then

$$\bar{\mathbf{x}}_{\epsilon_1} + \bar{\mathbf{x}}_{\epsilon_2} = (\epsilon_1 + \epsilon_2)\xi(\mathbf{x}) = \bar{\mathbf{x}}_{\epsilon_3} \in G, \quad (13)$$

and

$$\epsilon_3 = \epsilon_1 + \epsilon_2 \in \mathbb{R}. \quad (14)$$

2. **Identity**. If $\bar{\mathbf{x}}_0 \equiv I \in G$ such that for any $\epsilon \in \mathbb{R}$

$$\bar{\mathbf{x}}_0 + \bar{\mathbf{x}}_\epsilon = \bar{\mathbf{x}}_\epsilon = \bar{\mathbf{x}}_\epsilon + \bar{\mathbf{x}}_0, \quad (15)$$

then $\bar{\mathbf{x}}_0$ is an identity in G.

3. **Inverses**. For $\bar{\mathbf{x}}_\epsilon \in G$, $\epsilon \in \mathbb{R}$, there exists $\bar{\mathbf{x}}_\epsilon^{-1} \in G$, such that

$$\bar{\mathbf{x}}_\epsilon^{-1} + \bar{\mathbf{x}}_\epsilon = \bar{\mathbf{x}}_\epsilon + \bar{\mathbf{x}}_\epsilon^{-1}, \quad \bar{\mathbf{x}}_\epsilon^{-1} = \bar{\mathbf{x}}_{\epsilon^{-1}}, \quad (16)$$

and $\epsilon^{-1} = -\epsilon \in D$, where $+$ is a binary composition of transformations and it is understood that $\bar{\mathbf{x}}_\epsilon = \bar{\mathbf{x}}_\epsilon - \mathbf{x}$. Associativity follows from the closure property.

2.1.2 The Lie operator

For the multivariate function $\psi = \psi(x^1, \cdots, x^N; \epsilon)$, the expression (12) can be rewritten in the form

$$\bar{\mathbf{x}} = x + \epsilon \xi \frac{\partial x}{\partial x}, \quad (17)$$

or

$$\bar{x}^i = \left(1 + \epsilon \xi^i \frac{\partial}{\partial x^i}\right) x^i. \quad (18)$$

That is,

$$\bar{x}^i = (1 + \epsilon \xi \cdot \nabla) x^i, \quad (19)$$

where $\xi = (\xi^1, \cdots, \xi^N)$. That is,

$$\bar{x}^i = (1 + \epsilon X) x^i, \quad (20)$$

with

$$X = \sum_{i=1}^{N} \xi^i(x^1, \cdots, x^N) \frac{\partial}{\partial x^i}. \quad (21)$$

This operator is the symmetry generator.

2.1.3 Prolongations formulas

The operator X is not adequate generating symmetries for differential equations, where it applies. This, however, can be remedied through prolongations.

The case $N = 2$ has $x^1 = x$ and $x^2 = y$ so that

$$X = \xi(x, y) \frac{\partial}{\partial x} + \eta(x, y) \frac{\partial}{\partial y}. \quad (22)$$

In determining the prolongations, it is convenient to use the operator of total differentiation

$$D = \frac{\partial}{\partial x} + y' \frac{\partial}{\partial y} + y'' \frac{\partial}{\partial y'} + \cdots, \quad (23)$$

where

$$y' = \frac{dy}{dx}, \quad y'' = \frac{d^2 y}{dx^2}, \quad \cdots. \quad (24)$$

The derivatives of the transformed point is then

$$\bar{y}' = \frac{d\bar{y}}{d\bar{x}}. \quad (25)$$

Since

$$\bar{x} = x + \epsilon \xi \ \text{ and } \ \bar{y} = y + \epsilon \eta, \quad (26)$$

then

$$\bar{y}' = \frac{dy + \epsilon d\eta}{dx + \epsilon d\xi}. \tag{27}$$

That is,

$$\bar{y}' = \frac{dy/dx + \epsilon d\eta/dx}{dx/dx + \epsilon d\xi/dx}. \tag{28}$$

Now introducing the operator D:

$$\bar{y}' = \frac{y' + \epsilon D(\eta)}{1 + \epsilon D(\xi)} = \frac{(y' + \epsilon D(\eta))(1 - \epsilon D(\xi))}{1 - \epsilon^2 (D(\xi))^2}. \tag{29}$$

Hence

$$\bar{y}' = \frac{y' + \epsilon(D(\eta) - y'D(\xi)) - \epsilon^2 D(\xi)D(\eta)}{1 - \epsilon^2 (D(\xi))^2}. \tag{30}$$

That is,

$$\bar{y}' = y' + \epsilon(D(\eta) - y'D(\xi)), \tag{31}$$

or

$$\bar{y}' = y' + \epsilon\zeta^1, \tag{32}$$

with

$$\zeta^1 = D(\eta) - y'D(\xi). \tag{33}$$

It expands into

$$\zeta^1 = \left(\frac{\partial}{\partial x} + y'\frac{\partial}{\partial y}\right)\eta - y'\left(\frac{\partial}{\partial x} + y'\frac{\partial}{\partial y}\right)\xi, \tag{34}$$

so that

$$\zeta^1 = \eta_x + (\eta_y - \xi_x)y' - y'^2\xi_y. \tag{35}$$

The first prolongation of X is then

$$X^{[1]} = \xi(x,y)\frac{\partial}{\partial x} + \eta(x,y)\frac{\partial}{\partial y} + \zeta^1\frac{\partial}{\partial y'}. \tag{36}$$

For the second prolongation, we have

$$\bar{y}'' = \frac{y'' + \epsilon D(\zeta^1)}{1 + \epsilon D(\xi)} \approx y'' + \epsilon\zeta^2, \tag{37}$$

with

$$\zeta^2 = D(\zeta^1) - y''D(\xi). \tag{38}$$

This expands into

$$\begin{aligned} \zeta^2 &= \eta_{xx} + (2\eta_{xy} - \xi_{xx})y' + (\eta_{yy} - 2\xi_{xy})y'^2 \\ &- y'^3\xi_{yy} + (\eta_y - 2\xi_x - 3y'\xi_y)y''. \end{aligned} \tag{39}$$

The second prolongation of X is then

$$X^{[2]} = \xi(x,y)\frac{\partial}{\partial x} + \eta(x,y)\frac{\partial}{\partial y} + \zeta^1\frac{\partial}{\partial y'} + \zeta^2\frac{\partial}{\partial y''}. \tag{40}$$

2.1.4 Invariance

Theorem 2 *A function $F(x,y)$ is an invariant of the group of transformations if for each point (x,y) it is constant along the trajectory determined by the totality of transformed points $(\bar{x},\bar{y})$:*

$$F(\bar{x},\bar{y}) = F(x,y). \tag{41}$$

This requires that

$$XF = 0, \tag{42}$$

leading to the characteristic system

$$\frac{dx}{\xi} = \frac{dy}{\eta}. \tag{43}$$

Proof. Consider the Taylor series expansion of $F(\bar{\mathbf{x}})$ with respect to ϵ:

$$F(\bar{x},\bar{y}) = F(\bar{x},\bar{y})\Big|_{\epsilon=0} + \epsilon\frac{\partial\bar{F}}{\partial\epsilon}\Big|_{\epsilon=0} + \cdots. \tag{44}$$

This can be written in the form

$$F(\bar{x},\bar{y}) = F(\bar{x},\bar{y})\Big|_{\epsilon=0} + \epsilon\left(\frac{\partial\bar{x}}{\partial\epsilon}\frac{\partial\bar{F}}{\partial\bar{x}} + \frac{\partial\bar{y}}{\partial\epsilon}\frac{\partial\bar{F}}{\partial\bar{y}}\right)\Big|_{\epsilon=0} + \cdots. \tag{45}$$

That is,

$$F(\bar{x},\bar{y}) = F(x,y) + \epsilon\left(\xi\frac{\partial\bar{F}}{\partial\bar{x}} + \eta\frac{\partial\bar{F}}{\partial\bar{y}}\right)\Big|_{\epsilon=0} + \cdots, \tag{46}$$

or

$$F(\bar{x},\bar{y}) = F(x,y) + \epsilon\left(\xi\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial y}\right)\bar{F} + \cdots. \tag{47}$$

Hence

$$F(\bar{x},\bar{y}) = F(x,y) + \epsilon X\bar{F}, \tag{48}$$

with

$$X = \xi\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial y}. \tag{49}$$

This means if $X\bar{F} = 0$ we have

$$F(\bar{x},\bar{y}) = F(x,y), \tag{50}$$

which concludes the theorem.

2.2 Modified Symmetries

The modified one-parameter point symmetries and their properties reduce to the regular one-parameter point symmetries when $\omega \to 0$. This is an infinitesimal parameter that we shall introduce and associate with them.

2.2.1 One-Parameter Point Transformations

We build our discussion on smart symmetries from the following definition on . There could be could be some confusion because at some instances they seem to resemble , at other cases the one-parameter transformation view emerges. They also seem to be wedged between the two.

Definition 4 *Let*

$$\breve{\mathbf{x}} = \chi\left(\tilde{\mathbf{x}};\delta;\epsilon\right) \tag{51}$$

be a family of two-parameters $\{\epsilon,\delta\} \subset \mathrm{I\!R}$ *invertible transformations, of points* $\tilde{\mathbf{x}} = (\tilde{x}^1(x;\delta;\epsilon),\cdots,\tilde{x}^N(x;\delta;\epsilon)) \in \mathbf{R}^N$ *into points* $\breve{\mathbf{x}} = (\breve{x}^1,\cdots,\breve{x}^N) \in \mathbf{R^N}$. *These we call Neo one-parameter point transformations when subjected to the conditions*

$$\chi\left(\tilde{\mathbf{x}};\delta;\epsilon\right)\Big|_{\epsilon=0} = \tilde{\mathbf{x}}, \tag{52}$$

and

$$\tilde{\mathbf{x}}|_{\delta=0} = \mathbf{x}. \tag{53}$$

Furthermore,

$$\chi\left(\tilde{\mathbf{x}};\delta;\epsilon\right)\Big|_{\delta=0} = \bar{\mathbf{x}}, \tag{54}$$

so that

$$\chi\left(\tilde{\mathbf{x}};\delta;\epsilon\right)\Big|_{\delta=0,\epsilon=0} = \mathbf{x}, \tag{55}$$

for $\bar{\mathbf{x}} = (\bar{x}^1,\cdots,\bar{x}^N) \in \mathrm{I\!R}^N$ *and* $\mathbf{x} = (x^1,\cdots,x^N) \in \mathrm{I\!R}^\mathbf{N}$.

It should be obvious that these transformations are the regular two-parameter point transformations when the parameter both parameter ϵ and δ assume zero values. That is,

$$\chi\left(\tilde{\mathbf{x}};\delta;\epsilon\right)\Big|_{\epsilon=0,\delta=0} = \tilde{\mathbf{x}}, \tag{56}$$

or best expressed in the form

$$\chi\left(\tilde{\mathbf{x}};\delta;\epsilon\right)\Big|_{\epsilon=0,\delta=0} = \tilde{\mathbf{x}}. \tag{57}$$

They reduce to the one-parameter point transformations when the parameter δ is absent from the definition.

2.2.2 Modified local one-parameter group generators

In $\mathrm{I\!R}^2$, we have $\chi = (\phi;\psi)$, while $\breve{\mathbf{x}} = (\breve{x},\breve{y})$ and $\tilde{\mathbf{x}}(\delta) = (\tilde{x}(\delta);\tilde{y}(\delta))$, so that

$$\breve{\tilde{x}} = \phi\left(\tilde{x}(\delta),\tilde{y}(\delta),\epsilon\right) \tag{58}$$

and

$$\breve{y} = \psi\left(\tilde{x}(\delta),\tilde{y}(\delta),\epsilon\right) \tag{59}$$

Expanding (58) and (59) about $\epsilon = 0$, in some neighborhood of $\epsilon = 0$, gives

$$\breve{\tilde{x}} = \tilde{x}(\delta) + \epsilon\frac{\partial\tilde{G}}{\partial\epsilon}\bigg|_{\epsilon=0} + O\left(\epsilon^2\right). \tag{60}$$

That is,

$$\breve{\tilde{x}} = x+\delta\frac{\partial H}{\partial\epsilon}\bigg|_{\delta=0} +\epsilon\left(\frac{\partial G}{\partial\epsilon}\bigg|_{\delta=0,\epsilon=0} + \delta\frac{\partial^2 G}{\partial\epsilon\partial\delta}\bigg|_{\delta=0,\epsilon=0}\right). \tag{61}$$

This becomes

$$\breve{x} = x + \epsilon\frac{\partial G}{\partial\epsilon}\bigg|_{\delta=0,\epsilon=0} + \delta\frac{\partial H}{\partial\epsilon}\bigg|_{\delta=0}. \tag{62}$$

Letting

$$\xi = \frac{\partial G}{\partial\epsilon}\bigg|_{\delta=0,\epsilon=0} \tag{63}$$

and

$$\tilde{\xi} = \frac{\partial H}{\partial\epsilon}\bigg|_{\delta=0}, \tag{64}$$

gives the modified local one-parameter point transformation

$$\breve{\tilde{x}} = x + \epsilon\xi + \delta\tilde{\xi}, \tag{65}$$

leading to the symmetry generator

$$\tilde{X} = \sum_{i=1}^{N}\left(\xi^i + \frac{\delta}{\epsilon}\tilde{\xi}^i\right)\frac{\partial}{\partial x^i}, \tag{66}$$

It reduces to the regular generator (21) when $\delta = 0$. In the case where the ratio δ/ϵ assumes a finite complex value, as with $\delta = i\epsilon\omega$ with $\omega \in \mathrm{I\!R}$ being the finite value, then the operator is simply the complex symmetry generator,

$$\tilde{X} = \sum_{i=1}^{N}\left[\xi^i(x^1,\cdots,x^N) + i\omega\tilde{\xi}^i(x^1,\cdots,x^N;\omega)\right]\frac{\partial}{\partial x^i}, \tag{67}$$

otherwise it collapses into the regular symmetry generator.

2.2.3 Symmetry groups

An interesting property of symmetries $A = \{\tilde{X}_1, \tilde{X}_2, \cdots, \tilde{X}_n\}$ is that they also form a group, provided $\omega \to 0$. That is, the satisfy the following group properties:

1. **Closure**. If $\tilde{X}_1, \tilde{X}_2 \in A$, then
$$\tilde{X}_1 \circ \tilde{X}_2 = \tilde{X}_3 \in A.$$
2. **Identity**. If $\tilde{X}_0 \equiv I \in A$
$$\tilde{X}_0 \circ \tilde{X}_i = \tilde{X}_i = \tilde{X}_i \circ \tilde{X}_0, \quad i = 1, 2, \ldots, n$$
then $\tilde{X}_0$ is an identity in G.
3. **Inverses**. For $\tilde{X}_i \in G$, i=1,2, ...n, there exists $\tilde{X}_a^{-1} \in G$, such that
$$\tilde{X}_i^{-1} \circ \tilde{X}_i = \tilde{X}_i \circ \tilde{X}_i^{-1}$$
with
$$\tilde{X}_i^{-1} = \tilde{X}_{i^{-1}}, \quad i = 1, 2, \ldots n,$$

where $\circ$ is a . follows from the .

2.2.4 Invariance

Theorem 3 A function $F(\tilde{\mathbf{x}})$ is an invariant of the group of transformations if for each point $\tilde{\mathbf{x}}$ it is constant along the trajectory determined by the totality of transformed points $\breve{\tilde{\mathbf{x}}}$:

$$F(\breve{\mathbf{x}}) = F(\tilde{\mathbf{x}}). \tag{68}$$

This requires that

$$GF = 0, \tag{69}$$

leading to the characteristic system

$$\frac{d\tilde{x}^1}{\xi^1} = \cdots = \frac{d\tilde{x}^N}{\xi^N}. \tag{70}$$

Proof. Consider the of $F(\breve{\tilde{\mathbf{x}}})$ with respect to ϵ:

$$F(\breve{\mathbf{x}}) = F(\tilde{\mathbf{x}})\Big|_{\epsilon=0} + \epsilon \frac{\partial \breve{\mathbf{F}}}{\partial \epsilon}\Big|_{\epsilon=0} + \cdots. \tag{71}$$

This can be written in the form

$$F(\breve{\mathbf{x}}) = F(\breve{\mathbf{x}})\Big|_{\epsilon=0} + \epsilon \frac{\partial \breve{\mathbf{x}}}{\partial \epsilon} \cdot \bigtriangledown \breve{F}\Big|_{\epsilon=0} + \cdots. \tag{72}$$

That is,

$$F(\breve{\mathbf{x}}) = F(\breve{\mathbf{x}})\Big|_{\epsilon=0} + \epsilon\xi \cdot \bigtriangledown \breve{F}\Big|_{\epsilon=0} + \cdots. \tag{73}$$

For $\epsilon = 0$ then we get

$$F(\breve{\mathbf{x}}) = F(\tilde{\mathbf{x}})), \tag{74}$$

thus proving the theorem.

2.2.5 Prolongations formulas

Since

$$\tilde{x} = x + \epsilon\xi + \delta\tilde{\xi} \text{ and } \tilde{y} = y + \epsilon\eta + \delta\tilde{\eta}, \tag{75}$$

then

$$\tilde{y}' = \frac{dy + \epsilon d\eta + \delta d\tilde{\eta}}{dx + \epsilon d\xi + \delta d\tilde{\xi}}. \tag{76}$$

That is,

$$\tilde{y}' = \frac{dy/dx + \epsilon d\eta/dx + \delta d\tilde{\eta}/dx}{dx/dx + \epsilon d\xi/dx + \delta d\tilde{\xi}/dx}. \tag{77}$$

Now introducing the operator D:

$$\tilde{y}' = \frac{y' + \epsilon D(\eta) + \delta D\tilde{\eta}}{1 + \epsilon D\xi + \delta D\tilde{\xi}}. \tag{78}$$

Normalising the denominator:

$$\tilde{y}' = \left(\frac{y' + \epsilon D(\eta) + \delta D\tilde{\eta}}{1 + \epsilon D\xi + \delta D\tilde{\xi}}\right)\left(\frac{1 - \epsilon D\xi - \delta D\tilde{\xi}}{1 - \epsilon D\xi - \delta D\tilde{\xi}}\right). \tag{79}$$

$$\begin{aligned} \tilde{y}' &= \frac{y' + \epsilon[D(\eta) - y'D(\xi)] + \delta[D(\tilde{\eta}) - y'D(\tilde{\xi})]}{1 - \epsilon^2(D(\xi))^2 - \delta^2(D(\tilde{\xi}))^2 - 2\epsilon\delta(D\xi)(D\tilde{\xi})} \\ &+ \frac{-\epsilon^2 D(\xi)D(\eta) - \delta^2 D(\tilde{\xi})D(\tilde{\eta})}{1 - \epsilon^2(D(\xi))^2 - \delta^2(D(\tilde{\xi}))^2 - 2\epsilon\delta(D\xi)(D\tilde{\xi})}. \end{aligned} \tag{80}$$

$$\tilde{y}' = y' + \epsilon\left([D(\eta) - y'D(\xi)] + \omega[D(\tilde{\eta}) - y'D(\tilde{\xi})]\right). \tag{81}$$

That is,

$$\tilde{y}' = y' + \epsilon(D(\eta + \omega\tilde{\eta}) - y'D(\xi + \omega\tilde{\xi})), \tag{82}$$

or

$$\tilde{y}' = y' + \epsilon\tilde{\zeta}^1, \tag{83}$$

with

$$\tilde{\zeta}^1 = D(\eta + \omega\tilde{\eta}) - y'D(\xi + \omega\tilde{\xi}). \tag{84}$$

It expands into

$$\begin{aligned} \tilde{\zeta}^1 &= \left(\frac{\partial}{\partial x} + y'\frac{\partial}{\partial y}\right)(\eta + \omega\tilde{\eta}) \\ &-y'\left(\frac{\partial}{\partial x} + y'\frac{\partial}{\partial y}\right)(\xi + \omega\tilde{\xi}), \end{aligned} \tag{85}$$

so that

$$\tilde{\zeta}^1 = (\eta+\omega\tilde{\eta})_x + [(\eta+\omega\tilde{\eta})_y - \xi_x]\, y' - y'^2(\xi+\omega\tilde{\xi})_y. \tag{86}$$

The first prolongation of $\tilde{X}$ is then

$$\tilde{X}^{[1]} = \xi(x,y)\frac{\partial}{\partial x} + \eta(x,y)\frac{\partial}{\partial y} + \tilde{\zeta}^1\frac{\partial}{\partial y'}. \tag{87}$$

For the second prolongation, we note that since

$$\tilde{x} = x + \epsilon\xi + \delta\tilde{\xi} \quad \text{and} \quad \tilde{y}' = y' + \epsilon\tilde{\zeta}^1, \tag{88}$$

then

$$\bar{y}'' = \frac{y'' + \epsilon D(\tilde{\zeta}^1)}{1 + \epsilon D(\xi) + \sigma D(\tilde{\xi})}, \tag{89}$$

$$\bar{y}'' = \left(\frac{y'' + \epsilon D(\tilde{\zeta}^1)}{1 + \epsilon D(\xi) + \sigma D(\tilde{\xi})}\right)\left(\frac{1 - \epsilon D\xi - \delta D\tilde{\xi}}{1 - \epsilon D\xi - \delta D\tilde{\xi}}\right) \tag{90}$$

$$\bar{y}'' = \frac{\left(y'' + \epsilon D(\tilde{\zeta}^1)\right)\left(1 - \epsilon D\xi - \delta D\tilde{\xi}\right)}{1 - \epsilon^2(D(\xi))^2 - \delta^2(D(\tilde{\xi}))^2 - 2\epsilon\delta(D\xi)(D\tilde{\xi})}. \tag{91}$$

$$\bar{y}'' = \left(y'' + \epsilon D(\tilde{\zeta}^1)\right)\left(1 - \epsilon D\xi - \delta D\tilde{\xi}\right). \tag{92}$$

$$\bar{y}'' = y'' - \epsilon\left[D(\tilde{\zeta}^1) - y'' D(\xi + \omega\tilde{\xi})\right]. \tag{93}$$

with

$$\tilde{\zeta}^2 = D(\tilde{\zeta}^1) - y'' D(\xi + \omega\tilde{\xi}). \tag{94}$$

This expands into

$$\begin{aligned}\tilde{\zeta}^2 \;=\; & [\eta+\omega\tilde{\eta}]_{xx} + \left(2[\eta+\omega\tilde{\eta}]_{xy} - [\xi+\omega\tilde{\xi}]_{xx}\right) y' \\ & + \left([\eta+\omega\tilde{\eta}]_{yy} - 2[\xi+\omega\tilde{\xi}]_{xy}\right) y'^2 \\ & + \left([\eta+\omega\tilde{\eta}]_y - 2[\xi+\omega\tilde{\xi}]_x - 3y'[\xi+\omega\tilde{\xi}]_y\right) y'' \\ & - y'^3[\xi+\omega\tilde{\xi}]_{yy}.\end{aligned} \tag{95}$$

The second prolongation of $\tilde{X}$ is then

$$\tilde{X}^{[2]} = \xi(x,y)\frac{\partial}{\partial x} + \eta(x,y)\frac{\partial}{\partial y} + \tilde{\zeta}^1\frac{\partial}{\partial y'} + \tilde{\zeta}^2\frac{\partial}{\partial y''}. \tag{96}$$

2.3 A Simple Formula for Generating Modified Symmetries

The theory that we have just discussed in the preceeding section could be daunting to some. Fortunately, there is a simple procedure that can get one started. Consider the expression

$$bx + a, \tag{97}$$

that one usually encounters when investigating differential equations of the order two and above for symmetries. We will now show that it can be presented in the form

$$b\frac{\sin(i\omega[x + \frac{a}{b}])}{i\omega}, \tag{98}$$

for $\omega \to 0$. We will show later in the paper how it leads to the symmetries. In this section we concentrate on how it comes about.

2.3.1 Euler's ansatz

Leonhard Euler (1707–1783), investigated differential equations of the form

$$a_0\ddot{y} + b_0\dot{y} + c_0 y \;=\; 0, \tag{99}$$

using the ansatz

$$y = e^{\lambda x}, \tag{100}$$

for solutions. Here $y = y(x)$, with constant coefficients a_0, b_0 and c_0.

He concluded that

$$y = \begin{cases} e^{-\frac{b_0}{2a_0}x}\left(Ae^{-\tilde{\omega}x} + Be^{\tilde{\omega}x}\right), & b_0^2 > 4a_0c_0, \\ A + Bx, & b_0^2 = 4a_0c_0, \\ e^{-\frac{b_0}{2a_0}x}\left(A\cos(\tilde{\omega}x) + B\sin(\tilde{\omega}x)\right), & b_0^2 < 4a_0c_0, \end{cases} \tag{101}$$

where

$$\tilde{\omega} = \frac{\sqrt{b_0^2 - 4a_0c_0}}{2a_0}, \tag{102}$$

and A and B are constants.

That is, Euler determined three solution components: y_1 for $b_0^2 > 4a_0c_0$, y_2 for $b_0^2 = 4a_0c_0$ and y_3 for the case $b_0^2 < 4a_0c_0$.

These work well in practise and still find applications today, but they are mathematically unsound.

My belief is that he allow inconsistency to pass on based on the success of the formulas. Unfortunately, this has an enormous amount of work hinged on the error, and in some cases subsequently leading to cul de sacs.

2.3.2 Continuity issues

It is hard to believe that the great Euler did not notice the discontinuity in solutions. That is,

$$\lim_{\tilde{\omega}\to 0}(y_1 - y_2) \neq 0. \quad (103)$$

Also,

$$\lim_{\tilde{\omega}\to 0}(y_3 - y_2) \neq 0. \quad (104)$$

Maybe he may have thought this to be an inconsequential little shortcoming, but in practise these cases are always avoided consciously avoided because of the catastrophes that have arisen around them in the past. The collapse of the Tacoma narrows bridge is one example. Mathematically a lot of good can result from solving equation (99) exactly, such as what I am on about in this work.

2.3.3 An exact solution

To get an exact formula, first let

$$y = \beta z,$$

with $\beta = \beta(x)$ and $z = z(x)$, so that

$$\dot{y} = \dot{\beta} z + \beta \dot{z},$$

and

$$\ddot{y} = \ddot{\beta} z + 2\dot{\beta}\dot{z} + \beta\ddot{z}.$$

These transform (99) into

$$a_0\left(\ddot{\beta} z + 2\dot{\beta}\dot{z} + \beta\ddot{z}\right) + b_0\left(\dot{\beta} z + \beta\dot{z}\right) + c_0\beta z = 0.$$

That is,

$$a_0\beta\ddot{z} + \left(2a_0\dot{\beta} + b_0\beta\right)\dot{z} + \left(a_0\ddot{\beta} + b_0\dot{\beta} + c_0\beta\right) z = 0. \quad (105)$$

Choosing β to satisfy $2a_0\dot{\beta} + b_0\beta = 0$ simplifies equation (105). That is,

$$\beta = C_{00} e^{\frac{-b_0}{2a_0}x},$$

for some constant C_{00}. Equation (105) assumes the form

$$\ddot{z} = -\frac{a_0\ddot{\beta} + b_0\dot{\beta} + c_0\beta}{a_0\beta} z.$$

That is,

$$\ddot{z} = \left(\frac{b_0^2 - 4a_0c_0}{4a_0^2}\right) z.$$

But $\ddot{z}$ can be written as $\dot{z}\,dz/dx$. Therefore,

$$\dot{z}\frac{d\dot{z}}{dz} = \left(\frac{b_0^2 - 4a_0c_0}{4a_0^2}\right) z,$$

or

$$\dot{z}\,d\dot{z} = \left(\frac{b_0^2 - 4a_0c_0}{4a_0^2}\right) z\,dz.$$

That is,

$$\frac{\dot{z}^2}{2} = \left(\frac{b_0^2 - 4a_0c_0}{4a_0^2}\right)\frac{z^2}{2} + C_{01},$$

for some constant C_{01}. That is,

$$\dot{z} = \sqrt{\left(\frac{b_0^2 - 4a_0c_0}{4a_0^2}\right)\frac{z^2}{2} + C_{01}},$$

or

$$\frac{dz}{\sqrt{\left(\frac{b_0^2 - 4a_0c_0}{4a_0^2}\right) z^2 + 2C_{01}}} = dx.$$

That is,

$$\frac{dz}{\sqrt{A_{00}^2 - z^2}} = \sqrt{-\frac{b_0^2 - 4a_0c_0}{4a_0^2}}\, dx,$$

with $A_{00}^2 = 2C_{01}/\sqrt{-\frac{b_0^2-4a_0c_0}{4a_0^2}}$. Hence,

$$z = \frac{2C_{01}}{\sqrt{-\frac{b_0^2-4a_0c_0}{4a_0^2}}} \times \sin\left(\sqrt{-\frac{b_0^2 - 4a_0c_0}{4a_0^2}}\, x + C_{02}\right), \quad (106)$$

for some constant C_{02}. That is,

$$y = C_{00} e^{\frac{-b_0}{2a_0}x} \frac{2C_{01}}{\sqrt{-\frac{b_0^2-4a_0c_0}{4a_0^2}}} \times \sin\left(\sqrt{-\frac{b_0^2 - 4a_0c_0}{4a_0^2}}\, x + C_{02}\right). \quad (107)$$

Letting

$$\bar{\omega} = \sqrt{-\frac{b_0^2 - 4a_0c_0}{4a_0^2}}$$

we have

$y = C_{00} e^{\frac{-b_0}{2a_0}x}\ \frac{2C_{01}}{\bar{\omega}}\ \sin(\bar{\omega}\ x + C_{02}),$

or

$y \quad = \quad C_{00} e^{\frac{-b_0}{2a_0}x} \quad 2C_{01} \quad [\frac{\sin(C_{02})}{\bar{\omega}}\cos(\bar{\omega}\ x) \ +$

$\cos(C_{02}) \frac{\sin(\bar{\omega}\ x)}{\bar{\omega}}]$.
A reduction to the trivial case $\ddot{y} = 0$ requires that $\sin(C_{02}) = C_{03}\sin(\bar{\omega})$ and $\cos(C_{02}) = C_{04}\cos(\bar{\omega})$. That is, $C_{03}^2 + C_{04}^2 = 1$. Hence,
$y = C_{00} e^{\frac{-b_0}{2a_0}x}\ 2C_{01}\ [\frac{C_{03}\sin(\bar{\omega})}{\bar{\omega}}\cos(\bar{\omega}\ x) + C_{04}\cos(\bar{\omega}) \frac{\sin(\bar{\omega}\ x)}{\bar{\omega}}]$,
or simply

$$\begin{aligned} y &= C_{00} e^{\frac{-b_0}{2a_0}x}\ 2C_{01}\ \frac{C_{03}\sin(\bar{\omega})\cos(\bar{\omega}\ x)}{\bar{\omega}} \\ &+ C_{00} e^{\frac{-b_0}{2a_0}x}\ 2C_{01}\ \frac{C_{04}\sin(\bar{\omega}\ x)}{\bar{\omega}}. \end{aligned} \quad (108)$$

It is very vital to indicate that if the parameters $\bar{\omega}$ in the denominator and $\sin(\bar{\omega})$ are absorbed into the coefficients C_{01} and C_{03}, then formula (108) would reduce to one of Euler's formulas. But the consequences would be fatal, as formula (108) would not reduce to $y = A + Bx$ when $b_0 = c_0 = 0$, that is, when $\bar{\omega} = 0$.

2.3.4 The formula

The analysis of determining equations in symmetry analysis always involve equations of the form

$$\xi = bx + a, \quad (109)$$

similar to the second result in (101). The solution obtained in Section 2.3.3 suggests it can be written in the form

$$\xi = b\frac{\sin(i\omega[x + \frac{a}{b}])}{i\omega}, \quad (110)$$

subject to $\omega = 0$. This formula provides an easier way of generating modified symmetries.

3 A Lie group symmetrical approach

3.1 The classical approach

We seek here a continuous group of transformations for the equations (1), (2) and (3) through a generator

$$Y = \xi\frac{\partial}{\partial T} + \eta^1\frac{\partial}{\partial V} + \eta^2\frac{\partial}{\partial I} + \eta^3\frac{\partial}{\partial T}. \quad (111)$$

The operator $\tilde{Y}$, is the prolongation of Y and is

$$\tilde{Y} = Y + \zeta_1^1\frac{\partial}{\partial \dot{V}} + \zeta_1^2\frac{\partial}{\partial \dot{I}} + \zeta_1^3\frac{\partial}{\partial \dot{T}} \quad (112)$$

where

$$\begin{aligned} \zeta_1^1 &= D_t(\eta^1) - \dot{V}D_t(\xi), \\ \zeta_1^2 &= D_t(\eta^2) - \dot{I}D_t(\xi), \\ \zeta_1^3 &= D_t(\eta^3) - \dot{T}D_t(\xi). \end{aligned} \quad (113)$$

with the operators of total differentiation D_t, D_x, D_y and D_z given by

$$\begin{aligned} D_t &= \frac{\partial}{\partial t} + \dot{V}\frac{\partial}{\partial V} + \ddot{V}\frac{\partial}{\partial \dot{V}} \\ &+ \dot{I}\frac{\partial}{\partial I} + \ddot{I}\frac{\partial}{\partial \dot{I}} \\ &+ \dot{T}\frac{\partial}{\partial t} + \ddot{T}\frac{\partial}{\partial \dot{T}} + \cdots. \end{aligned} \quad (114)$$

This gives

$$\begin{aligned} \zeta_1^1 &= \eta_t^1 + \dot{V}\eta_V^1 + \dot{I}\eta_I^1 + \dot{T}\eta_T^1 \\ &\quad -\dot{V}(\xi_t + \dot{V}\xi_V + \dot{I}\xi_I + \dot{T}\xi_T) \quad (115) \\ \zeta_1^2 &= \eta_t^2 + \dot{V}\eta_V^2 + \dot{I}\eta_I^2 + \dot{T}\eta_T^2 \\ &\quad -\dot{I}(\xi_t + \dot{V}\xi_V + \dot{I}\xi_I + \dot{T}\xi_T), \quad (116) \\ \zeta_1^3 &= \eta_t^3 + \dot{V}\eta_V^3 + \dot{I}\eta_I^3 + \dot{T}\eta_T^3 \\ &\quad -\dot{T}(\xi_t + \dot{V}\xi_V + \dot{I}\xi_I + \dot{T}\xi_T). \quad (117) \end{aligned}$$

The first invariance condition gives

$$\begin{aligned} &\eta_t^1 + \dot{V}\eta_V^1 + \dot{I}\eta_I^1 + \dot{T}\eta_T^1 - \dot{V}(\xi_t + \dot{V}\xi_V + \dot{I}\xi_I + \dot{T}\xi_T) \\ &-p\eta^2 + c\eta^1 = 0, \end{aligned} \quad (118)$$

subjecting to the condition $\dot{V} = pI - cV$ separates into the monomials

$$\begin{aligned} 1 &: \eta_t^1 + \dot{V}(\eta_V^1 - \xi_t) - \dot{V}^2\xi_V \\ &\quad -p\eta^2 + c\eta^1 = 0, \quad (119) \\ \dot{I} &: \eta_I^1 - \dot{V}\xi_I = 0, \quad (120) \\ \dot{T} &: \eta_T^1 - \dot{V}\xi_T = 0. \quad (121) \end{aligned}$$

The second invariance condition gives

$$\begin{aligned} &\eta_t^2 + \dot{V}\eta_V^2 + \dot{I}\eta_I^2 + \dot{T}\eta_T^2 - \dot{I}(\xi_t + \dot{V}\xi_V + \dot{I}\xi_I + \dot{T}\xi_T) \\ &-\beta(T\eta^1 + V\eta^3) + \delta\eta^2 = 0, \end{aligned} \quad (122)$$

subjecting to the condition $\dot{I} = \beta TV - \delta I$ and it separates into the monomials

$$\begin{aligned} 1 &: \eta_t^2 + \dot{I}(\eta_I^2 - \xi_t) - \dot{I}^2\xi_I \\ &\quad -\beta(T\eta^1 + V\eta^3) + \delta\eta^2 = 0, \quad (123) \\ \dot{V} &: \eta_V^2 - \dot{I}\xi_V = 0, \quad (124) \\ \dot{T} &: \eta_T^2 - \dot{I}\xi_T = 0. \quad (125) \end{aligned}$$

The third invariance condition gives

$$\eta_t^3 + \dot{V}\eta_V^3 + \dot{I}\eta_I^3 + \dot{T}\eta_T^3 - \dot{T}(\xi_t + \dot{V}\xi_V + \dot{I}\xi_I + \dot{T}\xi_T) + \beta(T\eta^1 + V\eta^3) + d\eta^3 = 0, \quad (126)$$

subjecting to the condition $\dot{T} = [\lambda - \beta TV - dT]$ and it separates into the monomials

$$\begin{aligned} 1 \quad &: \eta_t^3 + \dot{T}(\eta_T^3 - \xi_t) - \dot{T}^2\xi_T \\ &\quad +\beta(T\eta^1 + V\eta^3) + d\eta^3 = 0, && (127)\\ \dot{V} \quad &: \eta_V^3 - \dot{T}\xi_V = 0, && (128)\\ \dot{I} \quad &: \eta_I^3 - \dot{T}\xi_I = 0. && (129) \end{aligned}$$

3.1.1 Analysis of the monomials

The determining equations lead to the following simplified forms

$$\xi_V = \xi_I = \xi_T = 0, \quad (130)$$

and

$$\begin{aligned} \eta_I^1 &= 0, && (131)\\ \eta_T^1 &= 0, && (132)\\ \eta_V^1 - \xi_t &= 0, && (133)\\ \eta_t^1 - p\eta^2 + c\eta^1 &= 0, && (134)\\ \eta_T^2 &= 0, && (135)\\ \eta_V^2 &= 0, && (136)\\ \eta_I^2 - \xi_t &= 0, && (137)\\ \eta_t^2 - \beta(T\eta^1 + V\eta^3) + \delta\eta^2 &= 0, && (138)\\ \eta_I^3 &= 0, && (139)\\ \eta_V^3 &= 0, && (140)\\ \eta_T^3 - \xi_t &= 0. && (141)\\ \eta_t^3 + \beta(T\eta^1 + V\eta^3) + d\eta^3 &= 0. && (142) \end{aligned}$$

these lead to the symmetries

$$\begin{aligned} Y_1 &= \frac{\partial}{\partial t}, && (143)\\ Y_2 &= e^{\delta t}\frac{\partial}{\partial I}. && (144) \end{aligned}$$

There is not much that can be done we these two symmetries in pursuit of solutions. This is why modified symmetries are essential.

3.2 Modified symmetries

The application of the formula (110) to the monomials (134), (138) and (142) lead to the infinitesimals

$$\begin{aligned} \xi &= i\omega\left(\tilde{D}_0 t + \tilde{D}_1\right), && (145)\\ \eta^1 &= \tilde{D}_0 \sin\left(i\omega\left[V + \frac{\tilde{A}_0 t + \tilde{A}_1}{\tilde{D}_0}\right]\right), && (146)\\ \eta^2 &= \tilde{D}_0 \sin\left(i\omega\left[I + \frac{\tilde{B}_0 t + \tilde{B}_1}{\tilde{D}_0}\right]\right), && (147)\\ \eta^3 &= \tilde{D}_0 \sin\left(i\omega\left[T + \frac{\tilde{C}_0 t + \tilde{C}_1}{\tilde{D}_0}\right]\right), && (148) \end{aligned}$$

where $\tilde{D}_0, \tilde{D}_1, \tilde{A}_0, \tilde{A}_1, \tilde{B}_0, \tilde{B}_1, \tilde{C}_0$ and $\tilde{C}_1$ are constant parameters.

That is,

$$\begin{aligned} \eta^1 &= i\omega\left(\tilde{A}_0 t + \tilde{A}_1\right)\cos(i\omega V) \\ &+ \tilde{D}_0 \cos\left(i\omega\left[\frac{\tilde{A}_0 t + \tilde{A}_1}{\tilde{D}_0}\right]\right) \\ &\times \sin(i\omega V), && (149)\\ \eta^2 &= i\omega\left(\tilde{B}_0 t + \tilde{B}_1\right)\cos(i\omega I) \\ &+ \tilde{D}_0 \cos\left(i\omega\left[\frac{\tilde{B}_0 t + \tilde{B}_1}{\tilde{D}_0}\right]\right) \\ &\times \sin(i\omega I), && (150)\\ \eta^3 &= i\omega\left(\tilde{C}_0 t + \tilde{C}_1\right)\cos(i\omega T) \\ &+ \tilde{D}_0 \cos\left(i\omega\left[\frac{\tilde{C}_0 t + \tilde{C}_1}{\tilde{D}_0}\right]\right) \\ &\times \sin(i\omega T), && (151) \end{aligned}$$

or simply

$$\begin{aligned} \eta^1 &= i\omega\left(\tilde{A}_0 t + \tilde{A}_1\right)\cos(i\omega V) \\ &\quad +\tilde{D}_0 \sin(i\omega V), && (152)\\ \eta^2 &= i\omega\left(\tilde{B}_0 t + \tilde{B}_1\right)\cos(i\omega I) \\ &\quad +\tilde{D}_0 \sin(i\omega I), && (153)\\ \eta^3 &= i\omega\left(\tilde{C}_0 t + \tilde{C}_1\right)\cos(i\omega T) \\ &\quad +\tilde{D}_0 \sin(i\omega T). && (154) \end{aligned}$$

These lead to the symmetries

$$\begin{aligned} Y_1 &= i\omega t\frac{\partial}{\partial t} + \sin(i\omega V)\frac{\partial}{\partial V} \\ &+ \sin(i\omega I)\frac{\partial}{\partial I} + \sin(i\omega T)\frac{\partial}{\partial T}, && (155)\\ Y_2 &= i\omega\frac{\partial}{\partial t}, && (156)\\ Y_3 &= i\omega\cos(i\omega V)\frac{\partial}{\partial V}, && (157) \end{aligned}$$

$$Y_4 = i\omega t \cos(i\omega V) \frac{\partial}{\partial V}, \tag{158}$$
$$Y_3 = i\omega \cos(i\omega I) \frac{\partial}{\partial I}, \tag{159}$$
$$Y_4 = i\omega t \cos(i\omega I) \frac{\partial}{\partial I}, \tag{160}$$
$$Y_5 = i\omega \cos(i\omega T) \frac{\partial}{\partial T}, \tag{161}$$
$$Y_6 = i\omega t \cos(i\omega T) \frac{\partial}{\partial T}. \tag{162}$$

The arguement used in determining the symmetries above was the knowledge that an expression of the form

$$\xi = a + tb, \tag{163}$$

can be rewritten in the form

$$\xi = \frac{a\phi \cos(\omega t/i) + b \sin(\omega t/i)}{\omega/i}. \tag{164}$$

The latter reduces to the former when $\omega = 0$. For more details on this, the reader is referred to [11] and [12].

3.3 Invariance solutions through Y_1

Prolongation of Y_1:

$$\zeta_1^1 = \dot{V}(\cos(i\omega V) - 1) \tag{165}$$
$$\zeta_1^2 = \dot{I}(\cos(i\omega I) - 1), \tag{166}$$
$$\zeta_1^3 = \dot{T}(\cos(i\omega T) - 1). \tag{167}$$

$$\frac{dt}{i\omega t} = \frac{dV}{\sin(i\omega V)} = \frac{d\dot{V}}{\dot{V}(\cos(i\omega V) - 1)} \tag{168}$$

$$u_1 = \frac{\tan\left(\frac{i\omega V}{2}\right)}{t}, \quad v_1 = \frac{i\omega \dot{V}}{[\cos\left(\frac{i\omega V}{2}\right)]^2} \tag{169}$$

$$\frac{dv_1}{du_1} = \frac{\frac{\omega^2 \sin\left(\frac{i\omega V}{2}\right)\dot{V}^2}{[\cos\left(\frac{i\omega V}{2}\right)]^3} + \frac{i\omega \ddot{V}}{[\cos\left(\frac{i\omega V}{2}\right)]^2}}{-\frac{\tan\left(\frac{i\omega V}{2}\right)}{t^2} + \frac{i\omega \dot{V} \sec^2\left(\frac{i\omega V}{2}\right)}{t}} \tag{170}$$

$$v_1 = \int \left\{ \frac{\frac{\omega^2 \sin\left(\frac{i\omega V}{2}\right)\dot{V}^2}{[\cos\left(\frac{i\omega V}{2}\right)]^3} + \frac{i\omega \ddot{V}}{[\cos\left(\frac{i\omega V}{2}\right)]^2}}{-\frac{\tan\left(\frac{i\omega V}{2}\right)}{t^2} + \frac{i\omega \dot{V} \sec^2\left(\frac{i\omega V}{2}\right)}{t}} \right\} du_1 + C. \tag{171}$$

This leads to

$$\lim_{\omega \to 0} \left\{ v_1 - \frac{du_1}{d\omega} \frac{\frac{\omega^2 \sin\left(\frac{i\omega V}{2}\right)\dot{V}^2}{[\cos\left(\frac{i\omega V}{2}\right)]^3} + \frac{i\omega \ddot{V}}{[\cos\left(\frac{i\omega V}{2}\right)]^2}}{-\frac{\tan\left(\frac{i\omega V}{2}\right)}{t^2} + \frac{i\omega \dot{V} \sec^2\left(\frac{i\omega V}{2}\right)}{t}} \right\} = C, \tag{172}$$

where the parameter C is an integrating constant.
Similarly,

$$\frac{dt}{i\omega t} = \frac{dI}{\sin(i\omega I)} = \frac{d\dot{I}}{\dot{I}(\cos(i\omega I) - 1)} \tag{173}$$

$$u_2 = \frac{\tan\left(\frac{i\omega I}{2}\right)}{t}, \quad v_2 = \frac{i\omega \dot{I}}{[\cos\left(\frac{i\omega I}{2}\right)]^2} \tag{174}$$

$$\lim_{\omega \to 0} \left\{ v_2 - \frac{du_2}{d\omega} \frac{\frac{\omega^2 \sin\left(\frac{i\omega I}{2}\right)\dot{I}^2}{[\cos\left(\frac{i\omega I}{2}\right)]^3} + \frac{i\omega \ddot{I}}{[\cos\left(\frac{i\omega I}{2}\right)]^2}}{-\frac{\tan\left(\frac{i\omega I}{2}\right)}{t^2} + \frac{i\omega \dot{I} \sec^2\left(\frac{i\omega I}{2}\right)}{t}} \right\} = C_2, \tag{175}$$

and

$$\frac{dt}{i\omega t} = \frac{dT}{\sin(i\omega T)} = \frac{d\dot{T}}{\dot{T}(\cos(i\omega T) - 1)} \tag{176}$$

$$u_3 = \frac{\tan\left(\frac{i\omega T}{2}\right)}{t}, \quad v_3 = \frac{i\omega \dot{T}}{[\cos\left(\frac{i\omega T}{2}\right)]^2} \tag{177}$$

$$\lim_{\omega \to 0} \left\{ v_3 - \frac{du_3}{d\omega} \frac{\frac{\omega^2 \sin\left(\frac{i\omega T}{2}\right)\dot{T}^2}{[\cos\left(\frac{i\omega T}{2}\right)]^3} + \frac{i\omega \ddot{T}}{[\cos\left(\frac{i\omega T}{2}\right)]^2}}{-\frac{\tan\left(\frac{i\omega T}{2}\right)}{t^2} + \frac{i\omega \dot{T} \sec^2\left(\frac{i\omega T}{2}\right)}{t}} \right\} = C_3. \tag{178}$$

The parameters C, C_2 and C_3 are integrating constants.

4 The extend variation of parameters approach

4.1 The theoretical basis

Let us begin by supposing that we are interested in the solution $f = f(z)$ of the differential equation

$$h(z, f(z), \dot{f}(z), \ddot{f}(z), \cdots, f^{(n)}(z)) = 0. \tag{179}$$

A power series expansion of f indicates that it has infinite zeroes. This assertion is fortified by the fundamental theorem of algebra, as proven, amongst many others, Joseph-Louis Lagrange (1736 – 1813). If we now where to suspend the fact that we are talking about zeroes, and see these discrete elements of set A. That is, $A = \{z_1, z_2, z_3, \cdots\}$, and $B = \{f(z_1), f(z_2), f(z_3), \cdots\}$. Interpolation theories maintain that the value $f(\xi)$ with $z_i < \xi < z_{i+1}$ can be determined without error, simply because A is an infinite set.

In particular, we note that the zeroes of f and $\ddot{f}$ through the L'hopital principle suggests that

$$f''(\bar{z})f'(\bar{z}) = f(\bar{z})f^{(3)}(\bar{z}) \tag{180}$$

Solving this expression generates what appears to be constants. They are constant parameters only in A, because they were determined in there. Elsewhere, like at $z = \xi$, they vary with z; similar to the method of variation of parameters, hence the name we chose. As a theorem, it can be expressed in the form

Theorem 4 *If $f = f(z)$ is defined on $\mathbf{R}$ and analytic on $\mathbf{D} \subset \mathbf{R}$, and has common zeros $\{z_1, z_2, z_3, \cdots\}$ with $\ddot{f}(z)$ in $\mathbf{D}$, then the differential equation*

$$F(z, f(z), \dot{f}(z), \cdots) = 0, \tag{181}$$

is compatible with

$$f^{(n)}(z)f^{(m+1)}(z) - f^{(m)}(z)f^{(n+1)}(z) = 0. \tag{182}$$

The proof follows through Lipschitz's boundedness conditions and L'Hopital's principle.

4.2 The solutions

Combining the equations (1), (2) and (3) into one gives

$$\begin{aligned}&\frac{c\delta\dot{V}}{p} - \frac{2\left(-\frac{2c}{p\beta} - \frac{2\delta}{p\beta}\right)\dot{V}^3}{V^3}\\&+\left(\frac{c}{p} + \frac{\delta}{p}\right)\ddot{V} + \frac{2\left(-\frac{2c}{p\beta} - \frac{2\delta}{p\beta}\right)\dot{V}\ddot{V}}{V^2}\\&-\frac{\left(\frac{2c}{p\beta} + \frac{2\delta}{p\beta}\right)\dot{V}\ddot{V}}{V^2}\\&+\left(\frac{1}{p} + \frac{\frac{2c}{p\beta} + \frac{2\delta}{p\beta}}{V}\right)V^{(3)} = 0,\end{aligned} \tag{183}$$

with I and T determined by

$$I = \frac{\dot{V} + cV}{p}, \tag{184}$$

and

$$T = \frac{\dot{I} + \delta I}{V}. \tag{185}$$

4.2.1 An Eulerian approach

To use the formula Leonhard Euler (1707–1783) introduced more than two centuries ago, but still popular today, in solving (180), we first express it in the form

$$\frac{V^{(3)}}{\ddot{V}} = \frac{\dot{V}}{V}. \tag{186}$$

It integrates into

$$\ddot{V} = CV, \tag{187}$$

where C is a constant. At this, Euler would require us to let

$$V = e^{rz}, \tag{188}$$

so that

$$\ddot{V} = re^{rz}. \tag{189}$$

These then lead to two solutions. Namely,

$$V_1 = e^{-\sqrt{C}z} \tag{190}$$

and

$$V_2 = e^{\sqrt{C}z}, \tag{191}$$

so that the general solution is

$$V = C_1V_1 + C_2V_2. \tag{192}$$

In the event $C = -\omega^2$, the two solutions have the form

$$V_3 = \cos(\omega z) \tag{193}$$

and

$$V_4 = \sin(\omega z), \tag{194}$$

so that

$$\tilde{V} = C_3V_3 + C_4V_4. \tag{195}$$

But there is something wrong with these solutions. For example, it should be logical that when $C = 0$ we should have

$$V = \tilde{V}. \tag{196}$$

Unfortunately this does not result. Hence our preference on the solution in the next subsection.

4.2.2 An exact solution to (187)

Equation (187) can be expressed in the form

$$\left(\frac{d\dot{V}}{dz}\right) dV = CVdV. \quad (197)$$

That is,

$$\left(\frac{dV}{dz}\right) d\dot{V} = CVdV, \quad (198)$$

or

$$\dot{V}d\dot{V} = CVdV. \quad (199)$$

Introducing the integral signs:

$$\int \dot{V}d\dot{V} = C\int VdV, \quad (200)$$

and it gives

$$\dot{V}^2 = CV^2 + B_1, \quad (201)$$

where B_1 is an integration constant. Integrating further, we first separate the variables

$$\frac{dV}{\sqrt{CV^2 + B_1}} = dz, \quad (202)$$

so that

$$\int \frac{dV}{\sqrt{V^2 + \frac{B_1}{C}}} = \int \sqrt{C}dz. \quad (203)$$

For $C = -\omega^2$, we have

$$\int \frac{dV}{\sqrt{\frac{B_1}{\omega^2} - V^2}} = \int -\omega dz, \quad (204)$$

or

$$\int \frac{d[\omega V/\sqrt{B_1}]}{\sqrt{1 - [\omega V/\sqrt{B_1}]^2}} = \int -\omega dz. \quad (205)$$

This gives

$$\mathrm{ArcSin}\left(\frac{\omega V}{\sqrt{B_1}}\right) = -\omega z - \phi \quad (206)$$

where ϕ is an integration constant. Hence,

$$V = a\frac{\sin(\omega z + \phi)}{\omega}, \quad (207)$$

for $\sqrt{B_1} = -a$. This is our exact solution, for which

$$V_1 = \frac{a\sin(\phi)\cos(\omega z)}{\omega}, \quad (208)$$

and

$$V_2 = -\frac{a\cos(\phi)\sin(\omega z)}{\omega}, \quad (209)$$

meaning the general solution is

$$V = C_1V_1 + C_2V_2. \quad (210)$$

We shall now demonstrate how a and ω can be determined through the solutions obtained through the Eulerian approach.

The parameters C_1 and C_2 follow from the initial or boundary conditions of the model. The quantities a and ω, on the other hand, are not necessarily constants. To determining them using (216), for example, we note that the roots of V are also at $\ddot{V} = 0$. This simply means within the set A, we have

$$\dot{V}_1 = -a\omega\sin(\omega z) = -a\omega, \quad (211)$$

and

$$V_1^{(3)} = a\omega^3\sin(\omega z) = a\omega^3. \quad (212)$$

Substituting (211) and (212) into (183) gives one of the equation that eventually determines a and ω. The second equation is obtained by differentiating (183).

Two valus of ω results

$$\omega_1 = \frac{ic + i\delta - Sqrt[(-ic - i\delta)^2 + 4c\delta]}{2} \quad (213)$$

and

$$\omega_3 = \frac{ic + i\delta + Sqrt[(-ic - i\delta)^2 + 4c\delta]}{2}. \quad (214)$$

The value of a corresponding to $\omega = \omega_1$ is

$$a = \frac{N}{D} \quad (215)$$

where

$N = 1/4(-((2c)/(p\beta)) - (2\delta)/(p\beta))(ic + i\delta - Sqrt[(-ic - i\delta)^2 + 4c\delta])^2 + 1/4((2c)/(p\beta) + (2\delta)/(p\beta))(ic + i\delta - Sqrt[(-ic - i\delta)^2 + 4c\delta])^2 + \lambda$

and

$D = (ic\delta)/p + 1/2(-(c/p) - \delta/p)(ic + i\delta - Sqrt[(-ic - i\delta)^2 + 4c\delta]) - (i(ic + i\delta - Sqrt[(-ic - i\delta)^2 + 4c\delta])^2)/(4p)$.

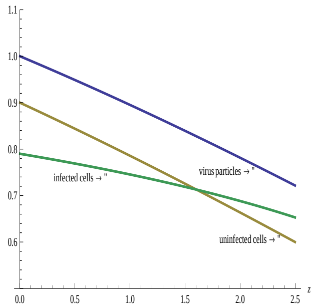

Figure 1: Plot of free virus particles (V), infected cells (I) and uninfected cells (T) against time (t), using results obtained through Euler's method.

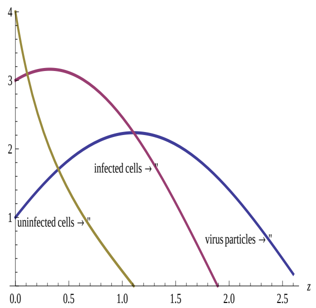

Figure 2: Plot of free virus particles (V), infected cells (I) and uninfected cells (T) against time (t), using exact results.

The solutions (216) and (217) follow from the well-known Euler ansatz of letting $V = \exp(rz)$, and then solving for r. It is not accurate, especially for small values of r. The exact results are

$$V_1 = a\omega \cos(\omega z), \tag{216}$$

and

$$V_2 = a\frac{\sin(\omega z)}{\omega}. \tag{217}$$

Euler's method is popular and widely accepted that it can't simply be wished away. We shall therefore plot both, and point out the contrasts.

5 Conclusions

The objective of the study was to solve a system of equations that explain blood dynamics in patients who had undergone an organ transplantation. The equations were found to be solvable through the tools we proposed.

To take our results to the level where they can find applications in medical care, we require to develop another model below what we already have. This secondary model will be expected to give interpretations at molecular and elements level, electro-dynamically.

The purpose for this further development would be remote sensing. A type of technology that would monitor the transplanted organs in patients from the comfort of their own homes, and at any time.

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