

Application of Fractal and Multifractal Analysis Algorithms to Image Segmentation and Classification

NATALIA AMPILOVA, IGOR SOLOVIEV

Comp. Sci. Dept.

St. Petersburg State University

University embankment, 7/9, St. Petersburg, 199034

RUSSIA

n.ampilova@spbu.ru i.soloviev@spbu.ru

Abstract: - Segmentation is one of image processing methods that results in partitioning a digital image into multiple parts (segments). This method may be used as a step in image analysis or may be considered as a single task. Segmentation is used to analyze medical images, faces, images of the Earth surface and many others. For complex structured images researchers in increasing frequency have to elaborate new methods of segmentation. In particular the application of classical segmentation methods to high resolution images often is not very successful, and fractal and multifractal methods seems to be more preferable, because they are more sensitive to pixel intensity changes. In this work we consider two methods of segmentation: calculation of fractal signatures for the elements of a partition of the surface formed by the image grey level function, and the method of multifractal spectrum calculation based on using density function. The both methods result in obtaining a set of numerical characteristics that may be used as classification and segmentation signs. The both methods show reliable results for images from Brodatz album and high resolution medical images of various types.

Keywords: - Fractal dimensions, fractal signature, multifractal spectrum, image segmentation and classification, density function, level sets.

1 Introduction

Segmentation, as one of main approaches to digital image analysis, is widely used in biomedical research. The applicability of a given segmentation method depends on the problem under investigation. The description of many classical segmentation algorithms is given in [10]. For magnetic resonance and ultrasonic images k-mean clustering, contour and edge detection methods are often applicable. So, in [7] when analyzing mammograms and selecting the regions of interest the authors used the segmentation method based on minimization of the active contour energy. Informative characteristics of the regions were considered as classification signs. Labeeb, Hamdy et al [11] used the mathematical morphology approach to define the regions of retina ischemia. The natural way to color images analysis is to obtain segmented images for all components of the palette, which results in identifying the color component giving significant segmentation performance. Such a technic was used in [1] to obtain the segmented red blood cells infected with malaria parasites. The common practice in image analysis is to apply several methods and compare results.

For images of complex structure researches tend

to use segmentation algorithms based on fractal and multifractal analysis. Such an approach is suitable for high resolution images that contain large number of fine details. Classical algorithms may not lead to reliable results because they often face into problems of specifying essential changes of pixel intensities.

Results of many observations and practical explorations show that for many objects and phenomena occurring in nature relations between their numerical characteristics satisfy the power law, which states that the estimated quantity (for example the area of square) is proportional to the side size in a power. For many geometric objects we observe a power law with integer degree. At the same time for real objects very often such a power is not integer but fractional. Thus we may operate with wide class of objects for that the relations between numerical characteristics may be expressed by using a fractional power. Such objects are called fractals or fractal sets. Fractal sets have a self-similarity property: a part of the image is similar to the whole one. Such a similarity may be strict (as the Cantor set or the Serpinsky carpet) or statistical. In many cases self-similarity of a set may be characterized by a scaling exponent (the degree in power law) which

is used to define the fractal dimension of the set. An in-depth analysis of connections between fractals and power laws and self-similarity may be found in [16].

There are various types of fractal dimensions. In practice the class of so called box-computing (box-counting) dimensions (e.g. capacity dimension) is widely used [6]. Let a set M be covered by $N(\varepsilon)$ boxes with a side ε . It is assumed that the number of boxes is proportional to $\varepsilon^{-\alpha}$. Decreasing ε and taking the limit of $\frac{\ln N(\varepsilon)}{\ln 1/\varepsilon}$ we obtain the value of α , which is called capacity dimension of M . It is easy to understand that box-computing dimensions may be calculated for binary images, because it is sufficient use only two colors to mark the belonging a point to the set. For greyscale (or monochrome) images the Minkovsky dimension, which coincides with capacity dimension for non-empty bounded sets in R^3 , may be used [6].

Multifractal set is a union of several fractals, which of them has its own fractal dimension. Components of multifractals are usually arranged by a complex interwoven way, hence segmentation problem by using classical methods is not very easy.

It is now widely accepted that many digital images may be considered as phase portraits of complex processes, especially images of biomedical preparations. Such images are complex, both regular and irregular structures. As a rule textures are often may be considered as fractal or multifractal sets [14]. Common assumption in image analysis of various textures is that the measure of an image (or its part) depends on the linear size, and this dependence may be expressed by a power law.

Image segmentation is a partitioning a digital image into multiple segments in accordance with a sign. More precisely, image segmentation is the process of assigning a label to every pixel in an image such that pixels with the same label share certain characteristics. The application both fractal and multifractal methods to image analysis results in obtaining the numerical characteristics that may be used as classifying signs. Besides that such characteristics may be considered as segmentation signs. That means that using fractal and multifractal analysis we may additionally obtain some kind of segmentation.

Fractal and multifractal methods are based on assumption that there is a measure defined in terms of pixel intensities. The image is divided on cells (boxes), for each cell its measure is calculated. The sum of pixel intensity is often used as a measure, but various types of filtration (Gaussian, Laplacian) are also applied [19]. The measure may be normed

depending on the method.

It is also assumed that the measure of i th cell $P_i(l) \sim l^{\alpha_i}$, where α_i are called singularity (scaling) exponents (power law again). One should calculate approximate value of these exponents and fractal dimensions of subsets that contain the cells having close values of exponents. The set of obtained fractal dimensions forms multifractal spectrum. To solve segmentation problem we have to find only exponents.

Segmentation based on using singularity exponents seems to be most appropriate for high resolution images. In [9, 18] the authors used Holder exponents to obtain segmentation of medical images.

In this work we consider two methods for image segmentation: fractal signature and multifractal analysis. The first method is based on the construction of the grey level function graphics for a greyscale image, which is a surface in 3-dimensional Euclidean space. For this surface we can calculate a characteristic called fractal signature and the Minkovsky dimension, which are simple related. The segmentation may be performed by partitioning of the image into cells and calculation fractal signature (or Minkovsky dimension) with the following marking the cells by a color in accordance with the obtained values. This method was described in [15] and successfully applied to analyze text documents [17], biomedical preparation images [2-4], and ISAR radar images [8, 12, 13]. One can obtain segmentation in the palette components.

The second method is the calculation of multifractal spectrum based on density function, which was supposed in [19]. For each point of an image a special characteristic — density function — is calculated. The image is divided on subsets of points that have close values of density function (so called level sets). Fractal dimensions of level sets form multifractal spectrum. This method was applied to obtain level sets for different biomedical images [5]. Any level set is a segmentation of an initial image.

We also consider a modification of the method and calculate density function for a small base cell, being the cell size is a parameter. Such an optimization reduces run time considerably and does not affect visibly on segmentation results. For both variants of the method it is density function value that is considered as segmentation sign.

Comparison of results of proposed methods and edge detection method in MATLAB showed that for 4 classes of high resolution biomedical preparation images fractal and multifractal methods are seemed to be more reliable.

2 Fractal Signature Method

The idea of the method is to calculate fractal dimension of the surface formed by the graphic of the grey level function for a given image. To do it we calculate the approximate value of the surface area, and this area may be obtained by using the volume of a special object.

Let $F = \{X_{ij}, i = 0, 1, \dots, K, j = 0, 1, \dots, L\}$ be an image and X_{ij} be the grey level of the (i, j) th pixel. In a certain measure range the grey-level surface of F can be viewed as a fractal. In image processing the grey level function F is a nonempty bounded set in R^3 . The surface area A_δ may be calculated using the volume of a special δ -parallel body — “blanket” with the thickness 2δ .

For $\delta = 1, 2, \dots$ the blanket surfaces are defined iteratively as follows:

$$u_\delta(i, j) = \max \left\{ \begin{array}{l} u_{\delta-1}(i, j) + 1, \\ \max_{|(m,n)-(i,j)| \leq 1} u_{\delta-1}(m, n) \end{array} \right\}$$

$$b_\delta(i, j) = \min \left\{ \begin{array}{l} b_{\delta-1}(i, j) - 1, \\ \min_{|(m,n)-(i,j)| \leq 1} b_{\delta-1}(m, n) \end{array} \right\}.$$

The volume of the blanket Vol_δ and A_δ computed as

$$Vol_\delta = \sum (u_\delta(i, j) - b_\delta(i, j))$$

$$A_\delta = \frac{Vol_\delta - Vol_{\delta-1}}{2} \quad (1)$$

The value $S_\delta = \frac{\log_2 A_\delta}{\log_2 \delta}$ is called fractal signature. Fractal dimension of the surface is defined by the formula

$$D \approx 2 - \frac{\log_2 A_\delta}{\log_2 \delta}. \quad (2)$$

It should be noted that we can use both fractal signatures and fractal dimensions as numerical characteristics for the following classification and segmentation.

In practice the approximate value of S_δ is obtained by the least square method as the slope of the line in axes $(\log_2 \delta, \log_2 A_\delta)$.

When solving segmentation problem we should use fractal signatures (or dimensions) for small parts of the given image — boxes (or cells). In this case boxes with similar characteristics form a segment. Hence the algorithm is the following. Partition the image into n cells by size $N \times N$ and then for $k = 1, 2, \dots, n$ calculate grey level functions F^k ,

u_0^k and b_0^k , u_δ^k and b_δ^k (for $\delta = 1, 2, 3$), Vol_δ^k and A_δ^k .

By the least square method calculate approximate values

$$S^k = \lim_{\delta \rightarrow 0} \frac{\log_2 A_\delta^k}{\log_2 \delta}.$$

Mark the cell having close values S^k by the same color.

We note that under such an approach the fractal signature method may be considered as a multifractal one: we obtain fractal dimensions of the partition cells, which is similar multifractal spectrum.

3 Multifractal Spectrum Computation

3.1 Base Method Description

Let μ be a measure defined through pixel intensities. For $x \in R^2$ we denote $B(x, r)$ a square of side r with center x . Let $\mu(B(x, r)) = kr^{d(x)}$, where $d(x)$ is the local density function of x , and k is some constant. Then

$$d(x) = \lim_{r \rightarrow 0} \frac{\log \mu(B(x, r))}{\log r}. \quad (3)$$

The density function measures the non-uniformity of the intensity distribution in the square $B(x, r)$. The set of all points x with local density α is a level set

$$E_\alpha = \{x \in R^2 : d(x) = \alpha\}.$$

The multifractal spectrum is defined as

$$\{f(\alpha) : \alpha \in R\} = \{\dim(E_\alpha) : \alpha \in R\}.$$

In practice, for not to increase the number of level sets, one can really consider sets

$$E(\alpha, \varepsilon) = \{x \in R^2, d(x) \in [\alpha, \alpha + \varepsilon]\}. \quad (4)$$

It is clear that showing level sets results in a segmentation of the image. Level sets are obtained in black and white palette: points from a level set are marked by black color. In this case capacity dimension is the appropriate variant of fractal dimension. Obtained level sets are shown for each α -interval separately. Vectors of fractal dimensions of level sets are used as classification signs for images from different classes.

3.2 Measure Choice

The measure $\mu(B(x,r))$ may be calculated by several ways. As it may depend on large illumination changes, the authors of [19] proposed to define the measure by using convolution operator with Gaussian smoothing kernel, diagonal and anti-diagonal directions in the square and the sum of Laplacians in the square. In this work we use two methods: the sum of intensity pixels in the square and the sum of Laplacians in the square.

1. $\mu(B(x,r))$ — the sum of intensity pixels in the square with side r and center x .

$$\mu(B(x,r)) = \iint I(y)dy,$$

where $I(y)$ — is the pixel intensity in $y \in B(x,r)$.

In discrete case

$$\mu(B(x,r)) = \sum_{y \in B(x,r)} I(y). \quad (5)$$

2. $\mu(B(x,r))$ — the sum of Laplacians in the square with side r and center x .

$$\mu(B(x,r)) = \iint \nabla^2(I(y))dy,$$

where ∇^2 — the Laplace operator.

In discrete case in accordance with [10] we have

$$\mu(B(x,r)) = \sum_{y \in B(x,r)} L(y), \quad (6)$$

where $L(y) = 8I(y) - \sum_{z \in B(y,2) \setminus y} I(z)$.

It was shown in [5] that namely using Laplacian resulted in the separation of classes of connective tissues in accordance with obtained spectra.

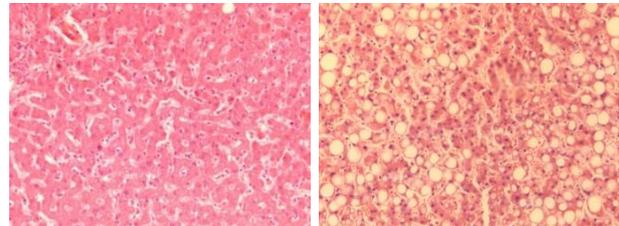
3.3 Application to Classification Problem

The density $d(x)$ is obtained as the slope of the line fitted to the data $\{\log r, \log \mu(B(x,r))\}$ by the least square method. Then we take a discrete set $\{\alpha_i\}$ from an interval (1,2) and find for each α_i the point set $E(\alpha_i, \varepsilon)$ ($\varepsilon = 0.1$) according to (4). This set contains all the pixels whose densities are close to α_i . Fractal dimensions $f(\alpha_i)$ may be computed by two methods: capacity dimension and the Minkovsky dimension.

Consider two classes of biomedical images — healthy liver and fatty liver disease. Each class contains 20 images. Color images were transformed to grey scale. Applying the described method we constructed level sets and calculated their fractal dimensions (both capacity and Minkovsky dimensions) for all the images. Then the average

multifractal spectrum for each class was calculated. The experiments were performed for two methods of the measure choice.

Samples of healthy liver and fatty liver disease images are shown on Fig.1.



a

b

Fig.1 Images of healthy liver (a) and fatty liver disease (b).

When choosing the measure of the square as the sum of pixel intensities we obtained graphics of average multifractal spectra (using Minkovsky dimension) which are shown on Fig.2.

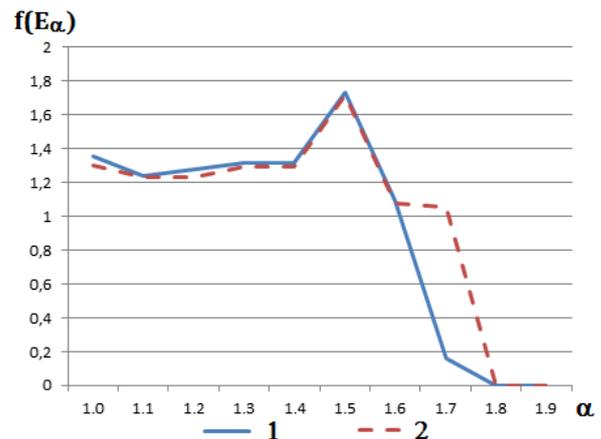


Fig.2 Graphics of average multifractal spectrum calculated by using Minkovsky dimension for two classes of liver images: 1— healthy liver, 2 — fatty liver disease. Box measure is the sum of pixel intensities.

Calculation of average multifractal spectrum with using capacity dimension lead to a similar result: average spectra for different classes are not separated.

At the same time calculations based on the measure with Laplacian allowed us to separate the graphics, as Fig.3 illustrates.

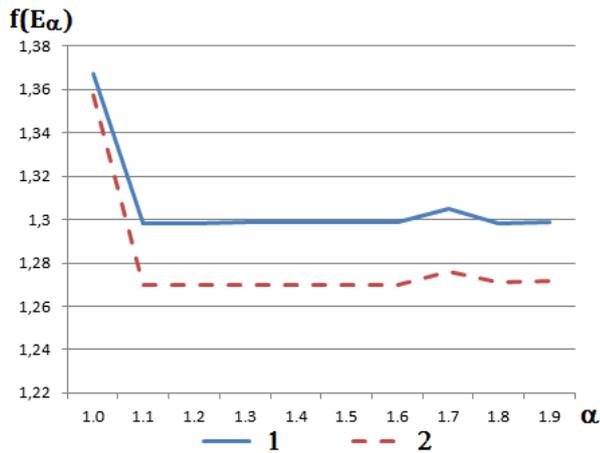


Fig.3 Graphics of average multifractal spectrum calculated by using Minkovsky dimension for two classes of liver images: 1— healthy liver, 2 — fatty liver disease. Box measure is calculated by using Laplacian.

3.4 Application to Segmentation Problem

It is easy to see that the level sets imaging is a kind of segmentation, where pixels are grouped in accordance with the sign defined density function values. Hence multifractal approach in addition to calculation of multifractal spectrum allows us to perform segmentation easily. In this case we obtain a set of segmented images. Thus, to obtain imaging for a level set $E(\alpha, \varepsilon)$ (we denote it $E(\alpha)$ for brevity) we should calculate density function for every pixel and form a new image, where pixels having density function values in interval $[\alpha, \alpha + \varepsilon)$ are marked by black color. For all obtained level sets we calculate their fractal dimensions and obtain multifractal spectrum. Such a characteristic may be considered as a classification sign for a class of images. It should be noted that results depend on the measure choice.

For the image of healthy liver (Fig. 1, a) we constructed level sets for two methods of the measure choice. When using the first method of measure calculation we obtain only two essential level sets — for $\alpha = 1.4$ and 1.6 . (Fig. 4) All the rest are practically empty.

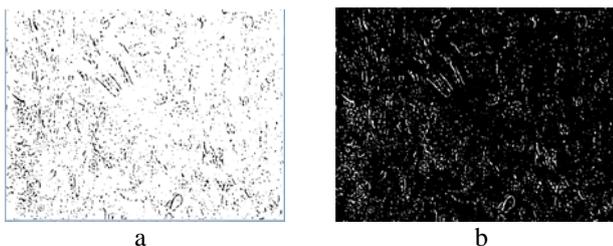


Fig.4 Level sets for the image of healthy liver: (a) — $\alpha = 1.4$, (b) $\alpha = 1.6$. Box measure is the sum of intensities.

When using the measure with Laplacian we obtain another distribution of density function and hence another collection of level sets. In this case the level sets containing more pixels are obtained for $\alpha = 1.1$ and $\alpha = 1.3$

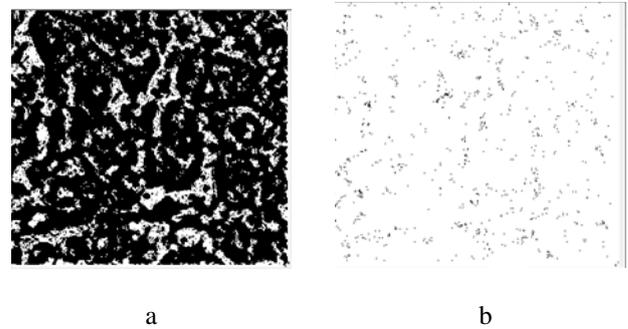


Fig.5 Level sets for the image of healthy liver: (a) — $\alpha = 1.1$, (b) $\alpha = 1.3$. Box measure is calculated by using Laplacian.

Thus, for image segmentation by using level sets the measure choice is inessential, because in any case we obtain a visual set of images characterizing an initial image.

It should be noted that the performance of the described algorithm is not very high: for the image with size 2500×2000 pixels run time required for construction of a level set is approximately 45 seconds. Hence to obtain several sets (in accordance with the 9 selected intervals of density function) we need more than 6 minutes.

4 Modification of the Base Method

It is easy to note that to obtain level sets we should use values of density function. Hence in segmentation problem we do not need to calculate multifractal spectrum. (For the classification problem both density function and multifractal spectrum are important numerical characteristics.) As the computing this function for every point of the image may be time-consuming we calculate it for a small base cell. As distinct from the base method we do not construct a set of binary level sets corresponding different intervals of density function value, but obtain one image in grayscale, where cells are colored in accordance with the value of its density function.

Let l be a side of a base cell, the measure (the sum of pixel intensities) of i th cell be $\mu_i(l)$, and density function — α_i .

In this case for segmentation we partition the image into n base cells by size $l \times l$ and for

$i = 1, 2, \dots, n$ calculate $\mu_i(l)$, select neighbourhoods of base cells by size $(l + 2) \times (l + 2)$ and $(l + 4) \times (l + 4)$, calculate $\mu_i(l + 2), \mu_i(l + 4)$. By the least square method using points $(\ln l, \ln \mu_i(l)), (\ln(l + 2), \ln \mu_i(l + 2)), (\ln(l + 4), \ln \mu_i(l + 4))$ calculate approximate values α_i . Mark base cells with close values of α_i by the same color.

The number of colors is selected by the user and the result depends on the range of exponent values.

5 Numerical experiments

The experiments were performed for images from Brodatz album [20] and biomedical preparations images including high resolution ones. For color images numerical characteristics were obtained for all components of RGB palette. The number of colors used for segmentation was selected in accordance with the range of fractal signatures (the method of fractal signature) and the range of density function for base cells (the method of multifractal spectrum calculation). To test the reliability of obtained results the method of edge detection from MATLAB package was applied. The results show that for complex textures considered methods are more preferable.

The table below shows the results of performance comparison for based and modified methods for different image sizes. In the base method the time of calculation of one level set is given. One should note that the modified method leads to better results both due to increasing the size of base cell and the construction only one segmented image instead of several level sets.

Image size Time (sec) \	428x431	640x640	2584x1936
Base method	26	31	45
Modified method	0.031	0.062	1.87

Table 1 Performance of based and modified methods

5.1 Brodaz Album

Grey scale images by size 640x640 were considered. Results of application of fractal signature method and modified method of calculation of density function are illustrated below. The side of partition box was selected equal 5 for both methods.

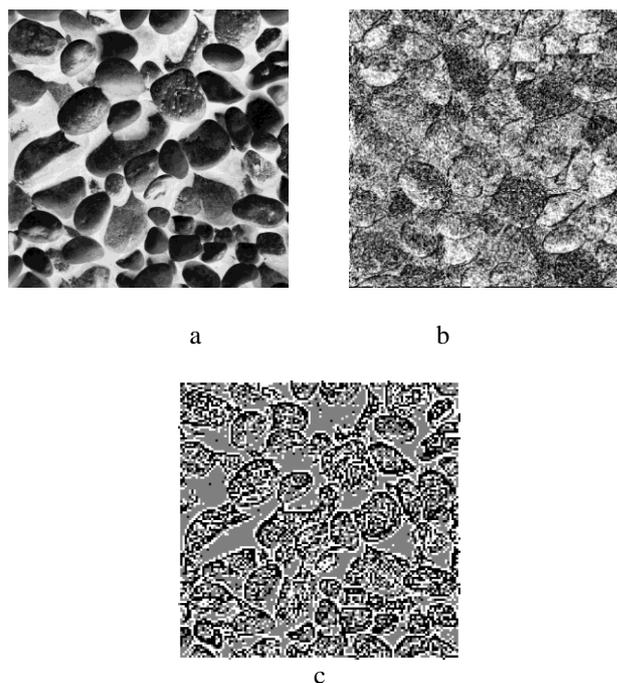


Fig. 6 Results of experiments: a — the initial image; b — application of fractal signature method ($l = 5$); c — application of multifractal method ($l = 5$).

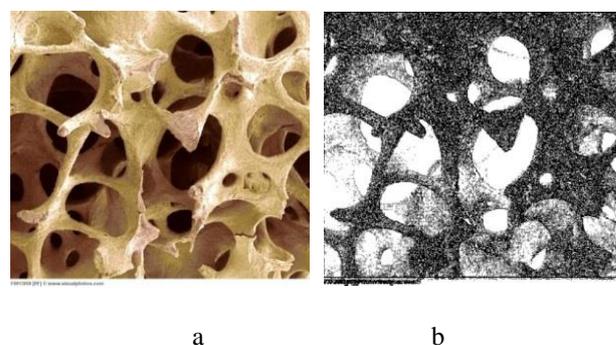
5.2 Bone Tissue with Osteoporosis

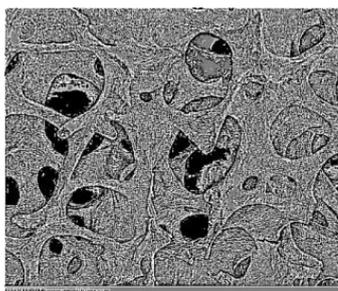
We applied our methods to images of bone tissue with osteoporosis. Initial images were segmented for different components of RGB palette and for greyscale variant of the initial image. The following illustrations show the best variant.

We note that the pictures c (results of application of the multifractal method) are shown in the larger scale than a and b for better illustrations of the method advantages.

5.2.1 G component of RGB

The image size is 700x606. The results of segmentation are shown for G component. The cell side is equal to 2, the number of colors for segmentation is 3.



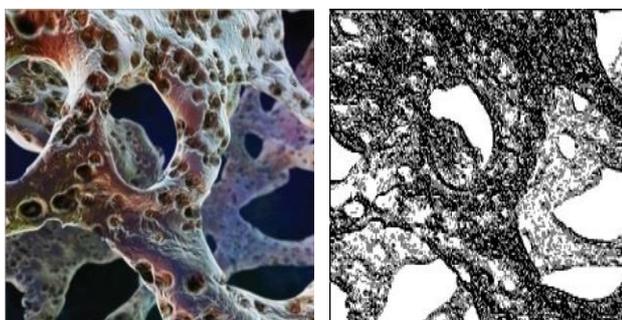


c

Fig.7 a — the initial image; b — application of fractal signature method ($l = 2$); c — application of multifractal method ($l = 2$).

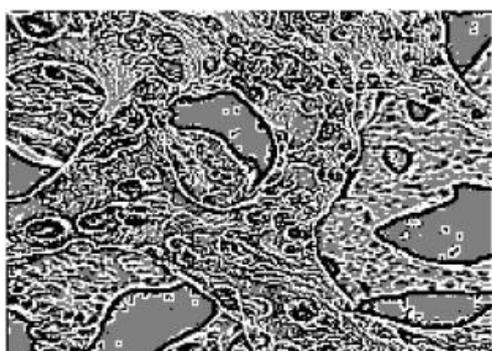
5.2.2 R component of RGB

The image size is 428x321. The results are shown for R component. The side of cell is equal to 2, the number of colors for segmentation equals 3.



a

b



c

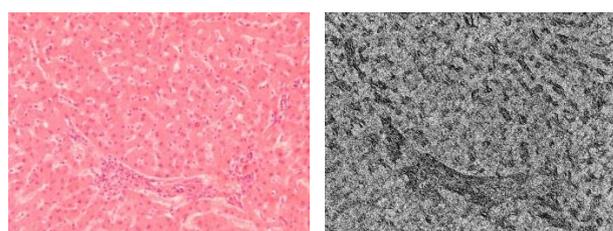
Fig.8 a — the initial image; b — application of fractal signature method ($l = 2$); c — application of multifractal method ($l = 2$).

5.3 High Resolution Images

We considered high resolution images of biomedical preparations (liver tissue) and compare the results with results of the edge detection method from MATLAB package. The images were obtained by microscope AxioCam MRc5 of the company Carl

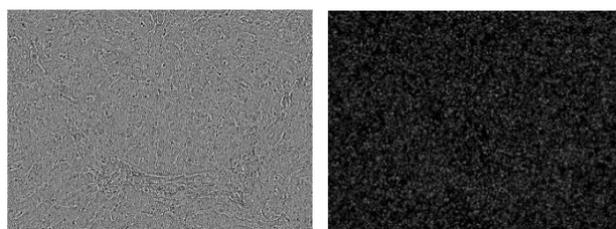
Zeiss Microimaging GmbH. All the images were made with 200-fold zoom, represented in RGB and have size 2584x1936 pixels. Experiments showed that for these images segmentation for G component of RGB is preferable. We show results of segmentation for the image of fatty liver disease and the image of liver with plethora. The illustrations demonstrate advantages of fractal methods for high resolutions images with complex structure comparing with edge detection method.

5.3.2 Liver with plethora



a

b



c

d

Fig.9 a — the initial image; b — application of fractal signature method ($l = 5$); c — application of multifractal method ($l = 5$); d — edge detection method (MATLAB).

6 Conclusion

Image segmentation may be obtained as an additional result of fractal or multifractal analysis. These methods allow us to calculate numerical characteristics that are input data for a classification problem. Moreover, the partitioning an image into parts in accordance with these data leads to easy segmentation algorithm. Run time for segmentation of high resolution images with large size is in average 1.5 sec, which makes possible their using in diagnostics. Experiments show that these methods are often more preferable for high resolution images than classical segmentation algorithms.

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