Transactions Briefs

A New Method for Computing the Stability Margin of Two-Dimensional Continuous Systems

Nikos E. Mastorakis and M. N. S. Swamy

***Abstract*—This paper presents a new method for computing the stability margin of two-dimensional (2-D) continuous systems. The method is based on the computation of the Hermite matrix in 2-D continuous systems, one of its partial derivatives and their resultant. The theoretical result is illustrated by examples.**

***Index Terms*—2-D continuous systems, stability margin.**

I. Introduction

Even in our digital computer era, continuous systems play a very important role in the development of modern electronic technology. In particular, two-dimensional (2-D) continuous systems ([1]–[10], [17], [30], [41]–[45]) have attracted the interest of many scientists and engineers for several reasons: In the design of 2-D and m-D $(m > 2)$ discrete filters, the corresponding analog filters play a dominant role. In particular, it is possible using appropriate transformations to obtain the desirable 2-D discrete filter from the corresponding analog (2-D) filter [2]–[9], [41]. On the other hand, in the study of distributed parameter systems which are described by partial differential equations (PDEs), each PDE actually corresponds to an m-D continuous system. So, for networks which include transmission lines as well as passive lumped elements, for networks containing semiconductor elements, for acoustic filters, the description with 2-D continuous systems is necessary as one can see in [1], [4], [7], [8]. A third reason is the need of the introduction of the 2-D continuous systems theory in control systems where the coefficients are functions of the parameters, as well as in systems whose inputs and outputs are functions of a time variable and a discrete spatial variable [8], [42]–[44]. Continuous models are also investigated in the so-called linear repetitive processes [46], [47]. For these reasons, there exists an importance of the subject of the m-D continuous systems from a practical point of view ([1]–[10], [17], [23], [30], [41]–[45]). Recently, in [48] two methods investigating the problem of stability margin computation for 2-D continous systems have been proposed. In the first method, an optimization problem has to be solved, whereas the second method is geometrical. In the present paper, a more "analytical" method based on an appropriate resultant computation will be presented.

It is known that a linear shift invariant 2-D continuous system can be described by the following transfer function:

$$G(s_1, s_2) = \frac{P(s_1, s_2)}{F(s_1, s_2)} \tag{1}$$

Manuscript received April 24, 2001; revised January 17, 2002. This paper was recommended by Co-Guest Editor S. Basu.

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Publisher Item Identifier S 1057-7122(02)05598-8.

where $P(s_1, s_2)$ and $F(s_1, s_2)$ are coprime polynomials in the independent complex variables s_1 and s_2, where we have assumed that there are no nonessential singularities of the second kind on the double imaginary axis, i.e., there are no points s_1, s_2 with $s_1 = j\omega_1$ or ∞, $s_2 = j\omega_2$ or ∞ such that $P(s_1, s_2) = F(s_1, s_2) = 0$.

The system (1) is bounded input bounded output (BIBO) stable (or equivalently Hurwitz stable) if and only if

$$F(1, s_2) \neq 0, \qquad \text{for} \quad \mathrm{Re}\{s_2\} \geq 0 \quad \text{or} \quad s_2 = \infty \tag{2.1}$$

and

$$F(s_1, j\omega_2) \neq 0, \qquad \text{for } (\mathrm{Re}\{s_1\} \geq 0 \text{ or } s_1 = \infty) \quad \text{and} \quad -\infty \leq \omega_2 \leq \infty. \tag{2.2}$$

Additionally, the polynomial $F(s_1, s_2)$ is said to be a BIBO *stable polynomial or a Hurwitz stable polynomial* if and only if (2.1) and (2.2) are fulfilled. Condition (2.1) is relatively easy to check using any one-dimensional (1-D) stability test. Checking condition (2.2) is a more difficult task.

Also, in [48], analogous to the definition of the *stability threshold* [23] or *stability margin* [31] for a 2-D discrete system, the following definitions have been recently introduced for a 2-D continuous system.

Definition 1: Given a 2-D continuous system described by the transfer function (1), we call stability margin σ_1 the greater non positive real number for which $F(s_1 + \sigma_1, s_2)$ is a *Hurwitz Polynomial*.

Definition 2: Given a 2-D continuous system described by the transfer function (1), we call stability margin σ_2 the greater non positive real number for which $F(s_1, s_2 + \sigma_2)$ is a *Hurwitz Polynomial*.

Definition 3: Given a 2-D continuous system described by the transfer function (1), we call stability margin σ the greater non positive real number for which $F(s_1 + \sigma, s_2 + \sigma)$ is a *Hurwitz Polynomial*.

II. Computation of the Stability Margins for 2-D Continuous Systems

Recently, in [48], two methods for the computation of the stability margin of 2-D continous systems have been proposed. In the first method, a constrained optimization problem has to be solved, whereas the second method is a geometrical method. A more "analytical" method based on an appropriate resultant computation will be developed here and we will test it on the same numerical examples of [48].

In [48], the following Proposition has been stated and used. Here, a more detailed proof is given.

Proposition 1: For the supremum of σ_1 for which $F(s_1 + \sigma_1, s_2)$ remains a *Hurwitz stable polynomial*, $\exists\, \omega_2$ with $-\infty \leq \omega_2 \leq \infty$ such that the Hermite matrix $H_1(\sigma_1, \omega_2)$ associated with $F(s_1 + \sigma_1, j\omega_2)$ is singular, i.e., $\det H_1(\sigma_1, \omega_2) = 0$.

Proof: Consider the mapping $h : \sigma_1 \to h(\sigma_1)$ where $h(\sigma_1) = H_1(\sigma_1, \omega_2)$ for which $F(s_1 + \sigma_1, s_2)$ remains a Hurwitz stable polynomial. This is a *continuous* mapping since the matrix $H_1(\sigma_1, \omega_2)$ is a two-variable polynomial in σ_1, ω_2. Also, the mapping $\det : h(\sigma_1) \to \det h(\sigma_1)$ is a continuous mapping and therefore, their synthesis $\det h\colon \sigma_1 \to \det h(\sigma_1)$ is also a continuous mapping. Let S be the set $S = \{h(\sigma_1) \text{ with } h(\sigma_1) > 0\}$, where $>$ denotes positive definite matrix $\forall \omega_2$ with $-\infty \leq \omega_2 \leq \infty$. We also denote $\det\{S\}$ the subset of the real numbers which consists of all the determinants of $h(\sigma_1)$ that belong to S. As one can see, 0 is the unique limit point for the set $\det\{S\}$.

S is an *open* set and because of the continuity of the mapping h, the corresponding set of σ_1 will also be *open* (see [11]). Thus, the supremum of σ_1, for which $F(s_1+\sigma_1, s_2)$ remains a Hurwitz stable polynomial, is a limit point of this set and because of the continuity of the mapping h, for this $\sigma_1, h(\sigma_1)$ is also a limit point of S. Furthermore, by the continuity of the mapping $\det : h(\sigma_1) \rightarrow \det h(\sigma_1), \det h(\sigma_1)$ is limit point in the set $\det\{S\}$, for this σ_1. Since the only limit point of $\det\{S\}$ is the 0, we conclude that for this σ_1, we have $\det h(\sigma_1) = 0$.

As a result, we obtain that for the supremum of σ_1 for which $F(s_1+\sigma_1, s_2)$ remains a *Hurwitz stable polynomial*, the Hermite matrix $H_1(\sigma, \omega_2)$ associated with $F(s_1+\sigma_1, j\omega_2)$ will be singular for some ω_2 with $-\infty \leq \omega_2 \leq \infty$. ■

Based on this Proposition, one can prove the following Theorem.

Theorem 1: For the supremum of σ_1 for which $F(s_1+\sigma_1, s_2)$ remains a Hurwitz stable polynomial, we have

$$\det H_1(\sigma_1, \omega_2) = 0 \tag{3}$$

and

$$\frac{\partial\{\det H_1(\sigma_1, \omega_2)\}}{\partial \omega_2} = 0 \tag{4}$$

for some real ω_2 $(-\infty \leq \omega_2 \leq \infty)$.

Proof: We have to prove only (4). Equation $\det H_1(\sigma_1, \omega_2) = 0$ defines a nonexplicit function of σ_1 in ω_2, i.e., $\sigma_1 = \sigma_1(\omega_2)$. Note that the supremum of σ_1 for which $F(s_1+\sigma_1, s_2)$ remains a Hurwitz stable polynomial, is simultaneously the infimum of σ_1 for which $\det H_1(\sigma_1, \omega_2) = 0$ has a solution, therefore is an infimum for the function $\sigma_1 = \sigma_1(\omega_2)$. So

$$\frac{\partial \sigma_1}{\partial \omega_2} = 0. \tag{5}$$

On the other hand, for σ_1 and ω_2 for which (3) holds, we have

$$\frac{\partial\{\det H_1(\sigma_1, \omega_2)\}}{\partial \sigma_1}\, d\sigma_1 + \frac{\partial\{\det H_1(\sigma_1, \omega_2)\}}{\partial \omega_2}\, d\omega_2 = 0. \tag{6}$$

Combining (5) and (6), (4) is easily proved. ■

Since (5) and (6) are polynomial equations with respect to ω_2, the existence of common roots is possible if and only if the resultant of $\det H_1(\sigma_1, \omega_2)$ and $(\partial\{\det H_1(\sigma_1, \omega_2)\})/\partial\omega_2$ (which will be a polynomial in σ_1) is 0. Therefore, the computation of σ_1 is achieved by the solution of

$$R\left\{\det H_1(\sigma_1, \omega_2), \frac{\partial\{\det H_1(\sigma_1, \omega_2)\}}{\partial \omega_2}\right\} = 0 \tag{7}$$

where $R\{\cdot, \cdot\}$ denotes the resultant of two polynomials.

The stability margin σ_2 can be determined by interchanging the indices 1 and 2.

Similar steps can lead to an analogous method for the stability margin σ. In this case, the polynomial $F(s_1+\sigma, s_2+\sigma)$ is considered. Similarly $F(s_1+\sigma, s_2+\sigma)$ remains Hurwitz stable if and only if the Hermite matrix $H(\sigma, \omega_2)$ associated with $F(s_1+\sigma, j\omega_2+\sigma)$ is positive definite for all ω_2 with $-\infty \leq \omega_2 \leq \infty$. Now, the following two theorems are proved in a similar manner to those of Proposition 1 and Theorem 1.

Proposition 2: For the supremum of σ for which $F(s_1+\sigma, s_2+\sigma)$ remains a Hurwitz stable *polynomial*, $\exists\, \omega_2$ with $-\infty \leq \omega_2 \leq \infty$ such that the Hermite matrix $H(\sigma, \omega_2)$ associated with $F(s_1+\sigma_1, j\omega_2)$ is singular i.e., $\det H(\sigma, \omega_2) = 0$.

Proof: Analogous to the proof of Proposition 1. ■

Theorem 2: For the supremum of σ for which $F(s_1+\sigma, s_2+\sigma)$ remains a Hurwitz stable polynomial, we have

$$\det H(\sigma, \omega_2) = 0 \tag{8}$$

and

$$\frac{\partial\{\det H(\sigma, \omega_2)\}}{\partial \omega_2} = 0 \tag{9}$$

for some real ω_2 $(-\infty \leq \omega_2 \leq \infty)$.

Proof: Analogous to the proof of Theorem 2. ■

Therefore, the stability margin σ is computed by the solution of

$$R\left\{\det H(\sigma, \omega_2), \frac{\partial\{\det H(\sigma, \omega_2)\}}{\partial \omega_2}\right\} = 0 \tag{10}$$

where $R\{\cdot, \cdot\}$ denotes the resultant of two polynomials.

Example 1: Consider the first-degree two-variable polynomial

$$F(s_1, s_2) = 7 + s_1 + 2s_2 + 2s_1 s_2. \tag{11}$$

It is always assumed that the corresponding 2-D system has no nonessential singularities of the second kind. Obviously, condition (2.1) holds while condition (2.2) can be easily checked via the positive definiteness of the Hermitian matrix which is $H_1(\omega_1) = 7 + 4\omega_2^2$. Therefore, $F(s_1, s_2)$ is Hurwitz stable polynomial. For the computation of the stability margin σ_1, one finds $\det H_1(\sigma_1, \omega_2) = (7+\sigma_1) + (4+4\sigma_1)\omega_2^2$. Therefore

$$\frac{\partial\{\det H_1(\sigma_1, \omega_2)\}}{\partial \omega_2} = (8 + 8\sigma_1)\omega_2.$$

So, their resultant is given as follows:

$$\begin{aligned} &R\left\{\det H_1(\sigma_1, \omega_2), \frac{\partial\{\det H_1(\sigma_1, \omega_2)\}}{\partial \omega_2}\right\} \\ &= \det \begin{bmatrix} 7+\sigma_1 & 0 & 4+4\sigma_1 \\ 0 & 0 & 8+8\sigma_1 \\ 0 & 8+8\sigma_1 & 0 \end{bmatrix} \\ &= -64(\sigma_1+1)^2(\sigma_1+7). \end{aligned}$$

Hence, the equation $R\{\det H_1(\sigma_1, \omega_2), (\partial\{\det H_1(\sigma_1, \omega_2)\}/\partial\omega_2)\} = 0$ yields $\sigma_1 = -1$, $\sigma_1 = -7$. Obviously, following the definition of the stability margin, the solution $\sigma_1 = -1$ is selected. Quite analogously, for the computation of the stability margin σ, one has

$$\det H(\sigma, \omega_2) = (7 + 3\sigma + 2\sigma^2)(1 + 2\sigma) + 4(1+\sigma)\omega_2^2.$$

Therefore

$$\frac{\partial\{\det H(\sigma, \omega_2)\}}{\partial \omega_2} = (8 + 8\sigma)\omega_2.$$

Hence, their resultant is given as follows:

$$\begin{aligned} &R\left\{\det H(\sigma, \omega_2), \frac{\partial\{\det H(\sigma, \omega_2)\}}{\partial \omega_2}\right\} \\ &= \det \begin{bmatrix} (7+3\sigma+2\sigma^2)(1+2\sigma) & 0 & 4+4\sigma \\ 0 & 0 & 8+8\sigma \\ 0 & 8+8\sigma & 0 \end{bmatrix} \\ &= -64(1+\sigma)^2(7+3\sigma+2\sigma^2)(1+2\sigma). \end{aligned}$$

Thus, the equation $R\{\det H_1(\sigma_1, \omega_2), (\partial\{\det H_1(\sigma_1, \omega_2)\}/\partial\omega_2)\} = 0$ yields $\sigma_1 = -1$, $\sigma_1 = -1/2$. Obviously, the solution $\sigma_1 = -1/2$ is selected.

Example 2: Let us consider the general first order characteristic polynomial of a 2-D (continuous) system of [48]

$$F(s_1, s_2) = 1 + as_1 + bs_2 + cs_1 s_2 \tag{12}$$

$$R\left\{\det H(\sigma,\omega_2), \frac{\partial\{\det H(\sigma,\omega_2)\}}{\partial\omega_2}\right\} = \det\begin{bmatrix}(1+a\sigma+b\sigma+c\sigma^2)(a+c\sigma) & 0 & c(b+c\sigma)\\ 0 & 0 & 2c(b+c\sigma)\\ 0 & 2c(b+c\sigma) & 0\end{bmatrix} = -4c^2(b+c\sigma)^2(1+a\sigma+b\sigma+c\sigma^2)(a+c\sigma)$$

$$\sigma = \begin{cases}\max\left(\frac{-(a+b)+\sqrt{(a+b)^2-4c}}{2}, -\frac{b}{c}, -\frac{a}{c}\right), & \text{if } (a+b)^2-4c \geq 0\\ \max\left(-\frac{b}{c}, -\frac{a}{c}\right), & \text{otherwise}\end{cases} \tag{15}$$

where $a, b, c > 0$. By verifying the same conditions as in the previous example, one can easily see that this is a Hurwitz stable polynomial. Similarly one finds

$$\det H_1(\sigma_1,\omega_2) = (1+a\sigma_1)a + (b+c\sigma_1)c\omega_2^2$$

and

$$\frac{\partial\{\det H_1(\sigma_1,\omega_2)\}}{\partial\omega_2} = 2(b+c\sigma_1)c\omega_2.$$

Therefore

$$R\left\{\det H_1(\sigma_1,\omega_2), \frac{\partial\{\det H_1(\sigma_1,\omega_2)\}}{\partial\omega_2}\right\} = \det\begin{bmatrix}(1+a\sigma_1)a & 0 & (b+c\sigma_1)c\\ 0 & 0 & 2(b+c\sigma_1)c\\ 0 & 2(b+c\sigma_1)c & 0\end{bmatrix} = -4ac^2(1+a\sigma_1)(b+c\sigma_1)^2.$$

Thus, the equation $R\{\det H_1(\sigma_1,\omega_2), (\partial\{\det H_1(\sigma_1,\omega_2)\}/\partial\omega_2)\} = 0$ yields

$$\sigma_1 = \max\left\{-\frac{b}{c}, -\frac{1}{a}\right\}. \tag{13}$$

By cyclic interchange of the indices, we obtain

$$\sigma_2 = \max\left\{-\frac{a}{c}, -\frac{1}{b}\right\}. \tag{14}$$

To obtain σ, we form the Hermitian determinant associated with $F(s_1+\sigma, j\omega_2+\sigma)$.

$$\det H(\sigma,\omega_2) = (1+a\sigma+b\sigma+c\sigma^2) \times (a+c\sigma) + c(b+c\sigma)\omega_2^2$$

and

$$\frac{\partial\{\det H(\sigma,\omega_2)\}}{\partial\omega_2} = 2c(b+c\sigma)\omega_2.$$

So, their resultant is given as the equation shown at the top of the page. Hence, we get (15) shown at the top of the page. One can notice the symmetry between a and b. So, we can verify the same result as in [48] using a simpler procedure. The above results can be extended to the m-D $(m > 2)$ continuous case after some simple modifications.

III. Conclusion

In this paper, a new method for computing the stability margin of 2-D continuous systems has been proposed. The method is based each time on the nullification of the resultant of an appropriate Hermite matrix and its derivative with respect to some frequency. The method seems to be better than two others recently published ones[48], since we are not obliged to solve a minimization problem as well as we avoid the geometrical approach.

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Two-Dimensional Analysis of an Iterative Nonlinear Optimal Control Algorithm

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***Abstract*—Nonlinear optimal control problems usually require solutions using iterative procedures and, hence, they fall naturally in the realm of 2-D systems where the two dimensions are response time horizon and iteration index, respectively. The paper uses this observation to employ 2-D systems theory, in the form of unit memory repetitive process techniques, to investigate optimality, local stability, and global convergence behavior of an algorithm, based on integrated-system optimization and parameter estimation, for solving continuous nonlinear dynamic optimal control problems. It is shown that 2-D systems theory can be usefully applied to analyze the properties of iterative procedures for solving these problems.**

***Index Terms*—2-D systems, convergence, optimal control, stability, unit memory repetitive processes.**

I. Introduction

The solution of nonlinear optimal control problems is often obtained in an iterative manner because of the existence of mixed boundary conditions. An algorithm is designed to update a trial solution from iteration to iteration. This falls naturally into the area of 2-D systems as defined in [1], where one dimension is the time horizon of the dynamic system under investigation and the other is the progress of the iterations. This was first recognized by Edwards and Owens [2] and then developed by Roberts [3] to employ linear 2-D system theory techniques to analyze local stability and convergence behavior of an algorithm known as DISOPE, which is an acronym for Dynamic System Optimization and Parameter Estimation. DISOPE is designed to achieve the solution of nonlinear optimal control problems subject to model-reality differences [4].

The 2-D analysis of optimal control is based on the theory of unit memory repetitive processes [5]. The optimal control application has been developed for discrete and continuous DISOPE algorithms and associated stability theorems have been obtained for local behavior about a limit profile when linear analysis is valid [3], [6]. An important result is that the resulting 2-D system contains an initial condition whose

Manuscript received April 25, 2001; revised January 11, 2002. This paper was recommended by Co-Guest Editor M. N. S. Swamy.

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Publisher Item Identifier S 1057-7122(02)05597-6.