[11] T. L. Vincent and J. Yu, "Control of a chaotic system," *Dyn. Control*, vol. 1, pp. 35–52, 1991.
[12] Y. Zeng and S. N. Singh, "Adaptive control of chaos in Lorenz system," *Dyn. Control*, vol. 7, pp. 143–154, 1997.
[13] W. Yu, "Passive equivalence of chaos in Lorenz system," *IEEE Trans. Circuits Syst. I*, vol. 46, pp. 876–878, July 1999.
[14] H. K. Khalil, *Nonlinear Systems*. Englwood Cliffs, NJ: Prentice Hall, 1996.
[15] M. Krstic, I. Kanellakopoulos, and P. Kokotovic, *Nonlinear and Adaptive Control*. New York: Wiley, 1995.
[16] J. S. Lin, "Nonlinear control design of discrete-time strict-feedback system," in *Prep. IFAC Symp. Control and Design*, Beijing, China, July 1999, pp. 557–562.
[17] D. L. Xu and S. R. Bishop, "A contraction-mapping-based control approach to stabilize chaotic systems," *Int. J. Bifurcation Chaos*, vol. 5, no. 6, pp. 1741–1748, 1995.
[18] K. Ogata, *Discrete-Time Control System*. Englewood Cliffs, NJ: Prentice-Hall, 1987.
[19] S. S. Ge, T. H. Lee, and C. Wang, "Adaptive backstepping control of a class of chaotic systems," in *Proc. 38th Conf. Decision Control*, Phoenix, Arizona, 1999, pp. 714–719.
[20] S. Mascolo and G. Grassi, "Controlling chaotic dynamics using backstepping design with application to the Lorenz system and Chua's circuit," *Int. J. Bifurcation Chaos*, vol. 9, no. 7, pp. 1425–1434, 1999.
[21] G. C. Goodwin and K. S. Sin, *Adaptive Filtering, Prediction and Control*. Englewood Cliffs, NJ: Prentice-Hall, 1984.
[22] P. C. Yeh and P. V. Kokotovic, "Adaptive control of a class of nonlinear discrete-time systems," *Int. J. Control*, vol. 62, no. 2, pp. 303–324, 1995.

A New Method for Computing the Stability Margin of 2-D Discrete Systems

Nikos E. Mastorakis

***Abstract*—This brief presents a new contribution in the problem of computing the stability margin of two-dimensional (2-D) discrete systems. The method, using the "resultant technique" instead of a typical minimization procedure , is actually an improvement of the method of .**

***Index Terms*—2-D Systems, multidimensional systems, stability, stability margin.**

I. Introduction

A single-input single-output, shift-invariant, causal two-dimensional (2-D) system can be described by the transfer function, $G(z_1, z_2) = (A(z_1, z_2))/(B(z_1, z_2))$ where $A(z_1, z_2)$ and $B(z_1, z_2)$ are coprime polynomials in the independent complex variables z_1 and z_2. It is assumed that there are no nonessential singularities of the second kind on the closed unit bidisk, i.e., there are no points (z_1, z_2) with $|z_1| \leq 1$ and $|z_2| \leq 1$ such that $A(z_1, z_2) = B(z_1, z_2) = 0$. In the study and design of 2-D systems, we are interested not only in whether the system is stable but also whether the system will remain stable in the presence of system parameter deviations.

For this reason, for a stable 2-D (discrete) system, the following definitions have been given [3], [9].

Manuscript received January 24, 2000; revised June 15, 2001. This paper was recommended by Editor M. N. S. Swamy.

The author is with the Military Institutions of University Education, Hellenic Naval Academy, Piraeus, Greece (e-mail: mastor@ieee.org).

Publisher Item Identifier S 1057-7122(01)09656-8.

Definition 1: Given a 2-D discrete system described by the transfer function $G(z_1, z_2)$, we call stability margin σ_1 the supremum (i.e., the lower upper bound) of the positive real numbers for which $B((1+\sigma_1) \cdot z_1, z_2)$ is a ***(Bounded Input Bounded Output, BIBO) Stable Polynomial***.

Definition 2: Given a 2-D discrete system described by $G(z_1, z_2)$, we call stability margin σ_2 the supremum of the positive real numbers for which $B(z_1, (1+\sigma_2) \cdot z_2)$ is a ***(BIBO) Stable Polynomial***.

Definition 3: Given a 2-D discrete system described by $G(z_1, z_2)$, we call stability margin σ the supremum of the positive real numbers for which $B((1+\sigma) \cdot z_1, (1+\sigma) \cdot z_2)$ is a ***(BIBO) Stable Polynomial***.

The concept of the margin of stability was originally due to Swamy, Roytman and Delansky who in their paper [2] discussed the effect of finite wordlength on the stability of multidimensional digital filters and defined the term "stability threshold", which later was redefined as "margin of stability" for 2-D filters [3].

Definition 4: Definition of the general stability margin σ with weights λ_1, λ_2 $(\lambda_1 + \lambda_2 = 1, \lambda_1 \geq 0$ and $\lambda_2 \geq 0)$: Given a 2-D discrete system described by the transfer function $G(z_1, z_2)$, we call stability margin $\sigma = \sigma(\lambda_1, \lambda_2)$ the supremum of the positive real numbers for which $B((1+\lambda_1\sigma) \cdot z_1, (1+\lambda_2\sigma) \cdot z_2)$ is a ***(BIBO) Stable Polynomial***.

It is also reminded that the system described by $G(z_1, z_2)$ as well as its characteristic polynomial $B(z_1, z_2)$ are called BIBO stable if and only if

$$B(0, z_2) \neq 0, \quad \text{for } |z_2| \leq 1 \tag{1}$$

$$B(z_1, z_2) \neq 0, \quad \text{for } |z_1| \leq 1, \quad |z_2| = 1. \tag{2}$$

Condition (1) requires a 1-D stability test while condition (2) requires a 2-D stability test. An overview of the various 2-D stability tests can be found in [1]. It is also known, that for the evaluation of the stability margin, several methods already exist [3]–[9].

II. A New Computational Method for the Stability Margins of a 2-D System

First, the following notation is used:

$$k_1 = 1 + \sigma_1. \tag{3}$$

For a stable 2-D discrete system, we recall that the polynomial $B(z_1, z_2)$ is a ***(BIBO) Stable Polynomial*** if and only if: (1) holds and the inners matrix $\Delta_{2N_1}(z_2)$ associated with $z_1^{N_1} B(z_1^{-1}, z_2)$ is positive innerwise for all $z_2, z_2 = e^{j\phi_2}$ and $\phi_2 \in [0, 2\pi]$, [10]. So, $B(k_1 z_1, z_2)$ remains ***(BIBO) Stable Polynomial*** if and only if (1) holds and the inners matrix $\Delta_{2N_1}(k_1, z_2)$ associated with $z_1^{N_1} B(k_1 z_1^{-1}, z_2)$ remains positive innerwise for all $z_2, z_2 = e^{j\phi_2}$ and $\phi_2 \in [0, 2\pi]$. However, because of the assumed stability of the considered system, (1) holds independent of k_1. (Note that (1) does not contain z_1, consequently it does not contain k_1.) Thus, $B(k_1 z_1, z_2)$ remains ***(BIBO) Stable Polynomial*** if and only if the inners matrix $\Delta_{2N_1}(k_1, z_2)$ associated with $z_1^{N_1} B(k_1 z_1^{-1}, z_2)$ remains positive innerwise for all $z_2, z_2 = e^{j\phi_2}$ and $\phi_2 \in [0, 2\pi]$.

Furthermore, if we consider the inner matrix $\Delta_{2N_1}(k_1, z_2)$ associated with $z_1^{N_1} B(k_1 z_1^{-1}, z_2)$, we obtain that for the supremum of k_1 for which $B(k_1 z_1, z_2)$ is ***(BIBO) Stable*** the inners matrix $\Delta_{2N_1}(k_1, z_2)$ will be singular i.e., $\det \Delta_{2N_1}(k_1, z_2) = 0$ (for some $z_2, z_2 = e^{j\phi_2}$ and $\phi_2 \in [0, 2\pi]$). A complete justification can be found in [9]. Therefore, the supremum of k_1 for which $B(k_1 z_1, z_2)$ is ***(BIBO) Stable*** is simultaneously the *minimum* of all k_1 with $\det \Delta_{2N_1}(k_1, z_2) = 0$ (for some $z_2, z_2 = e^{j\phi_2}$ and $\phi_2 \in [0, 2\pi]$).

The equation $\det \Delta_{2N_1}(k_1, z_2) = 0$ is a rational equation which its denominator is a power of z_2. Since $z_2 = e^{j\phi_2}$ i.e., $\neq 0$, this equation renders $A_1(k_1, z_2) = 0$ where $A_1(k_1, z_2)$ is the numerator of $\det \Delta_{2N_1}(k_1, z_2)$. The equation $A_1(k_1, z_2) = 0$ defines a function of k_1 with respect to z_2 if $(\partial A_1)/(\partial k_1) \neq 0$. Therefore, $(\partial k_1)/(\partial z_2) = -(\partial A_1/\partial z_2)/(\partial A_1/\partial k_1)$. For the minimum of k_1, we have $(\partial k_1)/(\partial z_2) = 0$ i.e., $(\partial A_1(k_1, z_2))/(\partial z_2) = 0$. Therefore, the minimum of k_1 fulfils simultaneously, the following equations:

$$A_1(k_1, z_2) = 0 \tag{4}$$

$$\frac{\partial A_1(k_1, z_2)}{\partial z_2} = 0 \tag{5}$$

where $A_1(k_1, z_2)$ is the numerator of $\det \Delta_{2N_1}(k_1, z_2) = 0$. Their common solution, with respect to k_1, can be found using the resultant of the above polynomials, [12]

$$R_{z_2}\left[A_1(k_1, z_2), \frac{\partial A_1(k_1, z_2)}{\partial z_2}\right] = 0. \tag{6}$$

Then, the stability margin σ_1 can be obtained from (3).

Remark 1: If $(\partial A_1)/(\partial k_1) = 0$ then, from $A_1(k_1, z_2) = 0$, we verify that $(\partial A_1)/(\partial z_2) = 0$ also. Therefore $A_1(k_1, z_2)$ is separable and can be written as follows $A_1(k_1, z_2) = A_{11}(k_1)A_{12}(z_2)$. Therefore, $\sigma_1 = k_1 - 1$, where k_1 is the minimum positive root k_1 of the equation $A_{11}(k_1) = 0$.

A similar method for the computation of σ_2 can be formulated by interchanging the roles of the variables z_1 and z_2. For the evaluation of the stability margin σ, one defines

$$k = 1 + \sigma. \tag{7}$$

For the stability margin σ, instead of (1) and (2), one uses the equivalent condition $B(z_1, z_2) \neq 0$, for $|z_1| \leq 1, |z_2| \leq 1$, [1]. So, k is the supremum of the real numbers (≥ 1) for which $B(kz_1, kz_2) \neq 0$, for $|z_1| \leq 1, |z_2| \leq 1$. Varying only z_2, one obtains that this condition is equivalent to $B(kz_1, kz_2) \neq 0$, for $|z_1| \leq 1, |z_2| = 1$. Following the same steps as in above, we have the equations

$$A(k, z_2) = 0 \tag{8}$$

$$\frac{\partial A(k, z_2)}{\partial z_2} = 0 \tag{9}$$

where $A(k, z_2)$ is the numerator of $\det \Delta_{2N_1}(k, z_2)$ and $\Delta_{2N_1}(k, z_2)$ is the inners matrix associated with $z_1^{N_1}B(kz_1^{-1}, kz_2)$. The common solution of (8) and (9), with respect to k, can be found using the resultant of the above polynomials, i.e.,

$$R_{z_2}\left[A(k, z_2), \frac{\partial A(k, z_2)}{\partial z_2}\right] = 0. \tag{10}$$

Afterwards, one can easily obtain σ from (7). To illustrate the proposed computational procedure, we consider the following example.

Remark 2: For the computation of σ_2 and σ, Remarks analogous to Remark 1 can be stated.

Example: The general first order characteristic polynomial of a stable system is considered. This example has also been investigated in [3]–[9].

$$B(z_1, z_2) = 1 + az_1 + bz_2 + cz_1z_2 \tag{11}$$

where a, b, c are real numbers. It is assumed that the corresponding 2-D system has no nonessential singularities of the second kind. To compute the stability margin σ_1, the inners matrix of $z_1^{N_1}B(k_1z_1^{-1}, z_2)$ is formed (here $N_1 = 1$). This matrix is

$$\Delta_{2N_1}(k_1, z_2) = \begin{bmatrix} (a + cz_2)k_1 & 1 + bz_2 \\ 1 + bz_2^{-1} & \left(a + cz_2^{-1}\right)k_1 \end{bmatrix} \tag{12}$$

where $z_2 = e^{j\phi_2}$ and $\phi_2 \in [0, 2\pi]$. Then

$$\det \Delta_{2N_1}(k_1, z_2) = \left(a^2 + c^2 + ac\frac{z_2^2 + 1}{z_2}\right)k_1^2 - \left(1 + b^2 + b\frac{z_2^2 + 1}{z_2}\right). \tag{13}$$

So, $A_1(k_1, z_2) = (ack_1^2 - b)z_2^2 + ((a^2 + c^2)k_1^2 - (1 + b^2))z_2 + (ack_1^2 - b)$ and $(\partial A_1(k_1, z_2))/(\partial z_2) = 2(ack_1^2 - b)z_2 + ((a^2 + c^2)k_1^2 - (1 + b^2))$. If we denote: $x = ack_1^2 - b$ and $y = (a^2 + c^2)k_1^2 - (1 + b^2)$ then

$$R_{z_2}\left[A_1(k_1, z_2), \frac{\partial A_1(k_1, z_2)}{\partial z_2}\right] = \det\begin{bmatrix} x & y & x \\ 0 & 2x & y \\ 2x & y & 0 \end{bmatrix} = 0$$

which finally yields $-y^2 + 4x^2 = 0$ from which one obtains $y = \pm 2x$. After simple algebraic manipulation, one can find that $k_1 = \min[|1 + b|/|a + c|, |1 - b|/|a - c|]$. From which

$$\sigma_1 = \min\left[\frac{|1 + b|}{|a + c|}, \frac{|1 - b|}{|a - c|}\right] - 1. \tag{14}$$

By symmetry of the polynomial (11), one evaluates

$$\sigma_2 = \min\left[\frac{|1 + a|}{|b + c|}, \frac{|1 - a|}{|b - c|}\right] - 1. \tag{15}$$

The results for the stability margins σ_1 and σ_2 agree with those in [3]–[9]. One also should note that the proposed method is simpler than that of [3]–[9]. In order to compute the third stability margin σ, we form the inners matrix of $z_1^{N_1}B(kz_1^{-1}, kz_2)$. This matrix is

$$\Delta_{2N_1}(k, z_2) = \begin{bmatrix} ak + ck^2z_2 & 1 + bkz_2 \\ 1 + bkz_2^{-1} & ak + ck^2z_2^{-1} \end{bmatrix}. \tag{16}$$

Therefore

$$\det \Delta_{2N_1}(k, z_2) = k^2(a^2 + c^2k^2 + ack(z_2^2 + 1)/(z_2)) - (1 + b^2k^2 + bk(z_2^2 + 1)/(z_2))$$

and

$$A(k, z_2) = (ack^3 - bk)z_2^2 + k^2((a^2 + c^2k^2) - (1 + b^2k^2))z_2 + (ack^3 - bk)$$

and

$$(\partial A(k, z_2))/(\partial z_2) = 2(ack^3 - bk)z_2 + k^2((a^2 + c^2k^2) - (1 + b^2k^2)).$$

If we denote $x = ack^3 - bk$ and $y = k^2((a^2 + c^2k^2) - (1 + b^2k^2))$ then

$$R_{z_2}\left[A(k, z_2), \frac{\partial A(k, z_2)}{\partial z_2}\right] = \begin{bmatrix} x & y & x \\ 0 & 2x & y \\ 2x & y & 0 \end{bmatrix} = 0$$

which finally yields $-y^2 + 4x^2 = 0$ from which we obtain $y = \pm 2x$. The latter equation renders

$$k^2(a^2 + c^2k^2 + 2ack) - (1 + b^2k^2 + 2bk) = 0 \tag{17}$$

and

$$k^2(a^2 + c^2k^2 - 2ack) - (1 + b^2k^2 - 2bk) = 0. \tag{18}$$

From (17) and (18) we find $k =$ minimum of the real positive values of the set

$$\left\{\frac{a+b\pm\sqrt{(a+b)^2-4c}}{2c}, \frac{-a+b\pm\sqrt{(-a+b)^2+4c}}{2c}, \frac{a-b\pm\sqrt{(a-b)^2-4c}}{2c}, \frac{-a-b\pm\sqrt{(-a-b)^2-4c}}{2c}\right\}.$$

Now, σ can be found using (7). The three stability margins agree with those in [3]–[9]. However, one has to note that here they derived in an easier manner avoiding any *minimization technique*.

III. Conclusion

For the margin of stability of 2-systems which was originally introduced in [2] many different methods have recently proposed [3]–[8]. In this brief, a new alternative method for the stability margin for 2-D discrete systems has been presented. The present method, using the resultant technique, has the advantage—compared with that of [9]—of avoiding the usual, typical and somewhat inconvenient minimization procedure which is used in [9].

Moreover, modifying the above method, one can easily derive a general algorithm for evaluating the general stability margin $\sigma = \sigma(\lambda_1, \lambda_2)$. Other recent results and methods related to 2-D system stability can be found in [14]–[32].

References

[1] E. I. Jury, *Multidimensional Systems (Techniques and Applications)*, S. G. Tzafestas, Ed. New York: Marcel Dekker, 1986.

[2] M. N. S. Swamy, L. M. Roytman, and J. F. Delansky, "Finite wordlength effect and stability of multidimensional digital filters," *Proc. IEEE*, vol. 69, Oct. 1981.

[3] P. Agathoklis, E. I. Jury, and M. Mansour, "The margin of stability of 2-D linear discrete systems," *IEEE Trans. Acoust., Speech, Signal Processing*, vol. ASSP-30, pp. 869–873, Dec. 1982.

[4] E. Walach and E. Zeheb, "N-dimensional stability margins computation and a variable transformation," *IEEE Trans. Acoust., Speech, Signal Processing*, vol. ASSP-30, pp. 887–893, Dec. 1982.

[5] W.-S. Lu, A. Antoniou, and P. Agathoklis, "Stability of 2-D digital filters under parameter variations," *IEEE Trans. Circuits Syst.*, vol. CAS-33, pp. 476–482, May 1986.

[6] L. M. Roytman, M. N. S. Swamy, and G. Eichmann, "An efficient numerical scheme to compute 2-D stability thresholds," *IEEE Trans. Circuits Syst.*, vol. CAS-34, pp. 322–324, Mar. 1987.

[7] M. Wang, E. B. Lee, and D. Booley, "A simple method to determine the stability and the margin of stability of 2-D recursive filters," *IEEE Trans. Circuits Syst. I*, vol. 39, pp. 237–239, Mar. 1992.

[8] D. Hertz and E. Zeheb, "Simplifications in multidimensional stability margin computations," *IEEE Trans. Acoust., Speech, Signal Processing*, vol. ASSP-35, pp. 566–568, Apr. 1987.

[9] N. Mastorakis, "A method of computing the 2-D stability margin," *IEEE Trans. Circuits Syst. II*, vol. 45, pp. 376–379, Mar. 1998.

[10] A. J. Kanellakis, S. G. Tzafestas, and N. J. Theodorou, "Stability Tests for 2-D systems using the Schwarz form and inners determinants," *IEEE Trans. Circuits Syst.* , vol. 9, pp. 1071–1077, Sept. 1991.

[11] E. I. Jury, "Modified stability table for 2-D digital filters," *IEEE Trans. Circuits Syst.* , pp. 116–119, Jan. 1988.

[12] ——, "The theory and applications of the inners," *Proc. IEEE*, vol. 63, pp. 1044–1068, July 1975.

[13] R. P. Roesser, "A discrete state-space model for image processing," *IEEE Trans. Automat. Contr.*, vol. AC-20, pp. 1–10, Feb. 1975.

[14] X. Yang, "Complex coefficients' table test for two-dimensional discrete systems," *J. North. Jiaotong Univ.*, vol. 18, no. 22, pp. 161–165, 1994.

[15] D. Raghuramireddy and R. Unbehaven, "Stability of a class of 2-D recursive filters without distortion of the amplitude spectrum," *IEEE Trans. Circuits Syst.*, vol. CAS-32, pp. 1296–1298, Dec. 1985.

[16] M. N. S. Swamy, L. M. Roytman, and N. Marinovic, "A necessary condition for the BIBO stability of 2-D filters," *IEEE Trans. Circuits Syst. II*, vol. 39, pp. 475–476, July 1992.

[17] J. Kurek and M. Zaremba, "Iterative learning control synthesis based on 2-D system theory," *IEEE Trans. Automat. Control*, vol. 38, pp. 121–125, Jan. 1993.

[18] W.-S. Lu, "On robust stability of 2-D discrete systems," *IEEE Trans. Automat. Control*, vol. 40, pp. 502–506, Mar. 1995.

[19] J. Kurek, "Stability of nonlinear parameter-varying digital 2-D systems," *IEEE Trans. Automat. Control*, vol. 40, pp. 1428–1432, Aug. 1995.

[20] T. Kaczorek, "Stabilization of singular 2-D continuous-discrete systems by state-feedback controllers," *IEEE Trans. Automat. Control*, vol. 41, pp. 1007–1009, July 1996.

[21] E. Rogers and D. H. Owens, *Stability Analysis for Linear Repetitive Processes*. New-York: Springer-Verlag, 1992.

[22] P. Agathoklis, E. Jury, and M. Mansour, "Algebraic necessary and sufficient conditions for the stability of 2-D discrete systems," *IEEE Trans. Circuits Syst. II*, vol. 40, pp. 251–258, Apr. 1993.

[23] J. Aravena, S. Masoud, and W. Porter, "State models and stability for 2-D filters," *IEEE Trans. Circuits Syst. II*, vol. 37, pp. 1509–1519, Dec. 1992.

[24] J. Aravena, "2-D BIBO stability using 1-D wave model," *IEEE Trans. Circuits Syst. I*, vol. 39, pp. 570–576, July 1992.

[25] P. Bauer and E. Jury, "BIBO-stability of multidimensional (m-D) shift-invariant discrete systems," *IEEE Trans. Automat. Control*, vol. 36, pp. 1057–1061, Sept. 1991.

[26] G. Gu and B. Lee, "A numerical algorithm for stability testing of 2-D recursive digital filters," *IEEE Trans. Circuits Syst.* , vol. 37, pp. 135–138, Jan. 1990.

[27] X. Hu and H. Yee, "Polynomial array for $F(z_1, z_2)$ on $|z_1| = 1$ and 2-D filter stability test," *IEEE Trans. Signal Processing*, vol. 40, pp. 1579–1581, June 1992.

[28] S. A. Yost and P. H. Bauer, "Shift-variant m-D systems and singularities on Tm: Implications for robust stability," *IEEE Trans. Circuits Syst. I*, vol. 42, pp. 477–479, Aug. 1995.

[29] L. Xu, O. Saito, and K. Abe, "Practical internal stability of nD discrete systems," *IEEE Trans. Automat. Control*, vol. 41, pp. 756–761, May 1996.

[30] K. Kurosawa, I. Yamada, and T. Yokokawa, "A fast 2-D stability test procedure based on FFT and its computational complexity," *IEEE Trans. Circuits Syst. II*, vol. 42, pp. 676–678, Oct. 1995.

[31] E. Curtin and S. Saba, "Stability and margin of stability tests for multidimensional filters," *IEEE Trans. Circuits Syst. I*, to be published.

[32] N. E. Mastorakis, "Comments concerning multidimensional polynomials properties," *IEEE Trans. Automat. Control*, vol. 41, p. 260, Feb. 1996.