Transactions Briefs

A Method for Computing the 2-D Stability Margin

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***Abstract*—In this brief, the margin of stability of two-dimensional (2-D) discrete systems is considered. A new method to compute the stability margin of 2-D continuous systems is provided. Illustrative examples are also included.**

***Index Terms*—Multidimensional systems, stability, stability margin, 2-D filters, 2-D systems.**

I. Introduction

Stability testing of the two-dimensional (2-D) and m-D $(m > 2)$ discrete systems is of much importance [1]. A shift-invariant causal single-input single-output 2-D system can be described by the transfer function

$$G(z_1, z_2) = \frac{A(z_1, z_2)}{B(z_1, z_2)} \tag{1}$$

where $A(z_1, z_2)$ and $B(z_1, z_2)$ are coprime polynomials in the independent complex variables z_1 and z_2. It is assumed that there are no nonessential singularities of the second kind on the closed unit bidisk, i.e., there are no points (z_1, z_2) with $|z_1| \leq 1$ and $|z_2| \leq 1$ such that

$$A(z_1, z_2) = B(z_1, z_2) = 0.$$

It is well known that system (1) is bounded input bounded output *(BIBO) Stable* if and only if

$$B(0, z_2) \neq 0, \qquad \text{for} \quad |z_2| \leq 1 \tag{2.1}$$

$$B(z_1, z_2) \neq 0, \qquad \text{for} \quad |z_1| \leq 1 \quad |z_2| = 1. \tag{2.2}$$

Condition (2.1) is relatively easy to check using any 1-D stability test. Condition (2.2) is more difficult since it includes two variables. We denote the following:

$$B(z_1, z_2) = \Sigma_{i_1=0}^{N_1} \Sigma_{i_2=0}^{N_2} b_{i_1, i_2} z_1^{i_1} z_2^{i_2}.$$

Additionally, the polynomial $B(z_1, z_2)$ is said to be (*BIBO*) *Stable* if and only if (2.1) and (2.2) are fulfilled.

There exist several algebraic methods for testing the stability of 2-D discrete systems or, equivalently, checking the BIBO character of 2-D polynomials [1].

In the study of 2-D systems, we are interested not only in whether the system is stable, but also whether the system will remain stable in the presence of system-parameter deviations.

For this reason, for a stable 2-D (discrete) system, the following definitions have been introduced [3]:

Definition 1: Given a 2-D discrete system described by the transfer function (1), we call stability margin σ_1 the supremum (i.e., the lower upper bound) of the positive real numbers for which $B((1 + \sigma_1) \cdot z_1, z_2)$ is a (BIBO) Stable Polynomial.

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Definition 2: Given a 2-D discrete system described by the transfer function (1), we call stability margin σ_2 the supremum of the positive real numbers for which

$$B(z_1, (1 + \sigma_2) \cdot z_2)$$

is a (BIBO) Stable Polynomial.

Definition 3: Given a 2-D discrete system described by the transfer function (1), we call stability margin σ the supremum of the positive real numbers for which

$$B((1 + \sigma) \cdot z_1, (1 + \sigma) \cdot z_2)$$

is a (BIBO) Stable Polynomial.

One should notice that the special case where the stable system has nonessential singularities of the second kind on the closed unit bidisk is excluded, since all three stability margins will be zero.

For the evaluation of the stability margin, several methods already exist [3]–[8]. In this brief, a new method is proposed. It is based on a recently proposed method for checking the stability of a 2-D system via inners determinants [9].

II. Computation of the Stability Margins for a 2-D (Discrete) System

In this paragraph, a method of computing the stability margins of 2-D systems is presented. First, we introduce the notation

$$k_1 = 1 + \sigma_1. \tag{3}$$

The method is based on checking the inners matrix of the characteristic polynomial $B(z_1, z_2)$ of a stable system described by (1). For a stable 2-D discrete system, we recall that the polynomial $B(z_1, z_2)$ is a (BIBO) Stable Polynomial if and only if (2.1) holds and the inners matrix

$$\Delta_{2N_1}(z_2)$$

associated with

$$z_1^{N_1} B(z_1^{-1}, z_2)$$

is positive innerwise for all

$$z_2, z_2 = e^{j\phi_2}$$

and $\phi_2 \in [0, 2\pi]$ [9]. Therefore,

$$B(k_1 z_1, z_2)$$

remains (BIBO) Stable Polynomial if and only if (2.1) holds and the inners matrix

$$\Delta_{2N_1}(k_1, z_2)$$

associated with

$$z_1^{N_1} B(k_1 z_1^{-1}, z_2)$$

remains positive innerwise for all

$$z_2, z_2 = e^{j\phi_2}$$

and

$$\phi_2 \in [0, 2\pi].$$

However, because of the assumed stability of the considered system, (2.1) holds independent of k_1. (Note that (2.1) does not contain z_1, consequently it does not contain k_1.) Thus, $B(k_1 z_1, z_2)$ remains a (BIBO) Stable Polynomial if and only if the inners matrix

$$\Delta_{2N_1}(k_1, z_2)$$

associated with

$$z_1^{N_1} B(k_1 z_1^{-1}, z_2)$$

remains positive innerwise for all

$$z_2, z_2 = e^{j\phi_2}$$

and

$$\phi_2 \in [0, 2\pi].$$

Considering the inners matrix

$$\Delta_{2N_1}(k_1, z_2)$$

associated with $z_1^{N_1} B(k_1 z_1^{-1}, z_2)$, we obtain that for the supremum of k_1 for which $B(k_1 z_1, z_2)$ is (BIBO) Stable the inners matrix

$$\Delta_{2N_1}(k_1, z_2)$$

will be singular, i.e.,

$$\det \Delta_{2N_1}(k_1, z_2) = 0$$

(for some $z_2, z_2 = e^{j\phi_2}$ and $\phi_2 \in [0, 2\pi]$). For a detailed justification, see the Appendix. Therefore, the supremum of k_1 for which $B(k_1 z_1, z_2)$ is (BIBO) Stable is simultaneously the *minimum* of all k_1 with

$$\det \Delta_{2N_1}(k_1, z_2) = 0$$

(for some $z_2, z_2 = e^{j\phi_2}$ and $\phi_2 \in [0, 2\pi]$). This implies that the computation of k_1 can be achieved by solving the following minimization problem:

$$\min k_1 \tag{4.1}$$

under the constraint

$$\det \Delta_{2N_1}(k_1, z_2) = 0 \tag{4.2}$$

where

$$\Delta_{2N_1}(k_1, z_2)$$

is the inners matrix associated with

$$z_1^{N_1} B(k_1 z_1^{-1}, z_2)$$

In the sequel, we easily obtain σ_1 from (3).

By interchanging the roles of the variables z_1 and z_2, a completely analogous method for the computation of σ_2 is obtained.

Analogously, for the computation of σ, we denote

$$k = 1 + \sigma. \tag{5}$$

Here, instead of (2.1) and (2.2), we use the equivalent condition [1]

$$B(z_1, z_2) \neq 0$$

for $|z_1| \leq 1, |z_2| \leq 1$ Thus, k is the supremum of the real numbers (≥ 1) for which

$$B(kz_1, kz_2) \neq 0, \qquad \text{for } |z_1| \leq 1, |z_2| \leq 1$$

Varying only z_2, one can obtain that this condition is equivalent to

$$B(kz_1, kz_2) \neq 0, \qquad \text{for } |z_1| \leq 1, |z_2| = 1$$

This latter equation is analogous to (2.2). Therefore, following exactly the same steps as in the case of the stability margin σ_1, we formulate the following method for the stability margin σ:

$$\min k \tag{6.1}$$

under the constraint

$$\det \Delta_{2N_1}(k, z_2) = 0 \tag{6.2}$$

where $\Delta_{2N_1}(k,_2)$ is the inners matrix associated with $z_1^{N_1} B(kz_1^{-1}, kz_2)$. The following example illustrates the implementation of this method.

Example 1 [3]–[8]: Consider the general first-order characteristic polynomial of a stable system

$$B(z_1, z_2) = 1 + az_1 + bz_2 + cz_1 z_2 \tag{7}$$

where a, b, c are real numbers. It is always assumed that the corresponding 2-D system has no nonessential singularities of the second kind. For the computation of the stability margin σ_1, one forms the inners matrix of $z_1^{N_1} B(k_1 z_1^{-1}, z_2)$ (here, $N_1 = 1$). This is

$$\Delta_{2N_1}(k_1, z_2) = \begin{bmatrix} (a + cz_2)k_1 & 1 + bz_2 \\ 1 + b\overline{z}_2 & (a + c\overline{z}_2)k_1 \end{bmatrix} \tag{8}$$

where $z_2 = e^{j\phi_2}$ and $\phi_2 \in [0, 2\pi]$ and the overbar denotes a complex conjugate. Then

$$\det \Delta_{2N_1}(k_1, z_2) = (a^2 + c^2 + 2acx)k_1^2 - (1 + b^2 + 2bx) \tag{9}$$

where $x = \cos\phi_2 (x \in [-1, 1])$. One obtains that $\det \Delta_{2N_1}(k_1, z_2)$ is linear in x. Thus, for a certain k_1, the minimum value of $\det \Delta_{2N_1}(k_1, z_2)$ is obtained for $x = -1$ (if $ack_1^2 - b \geq 0$) or for $x = +1$ (if $ack_1^2 - b < 0$). Thus, for the *minimum* k_1 with $\det \Delta_{2N_1}(k_1, z_2) = 0$, the determinant $\det \Delta_{2N_1}(k_1, z_2)$ will be zero for $x = \pm 1$. Therefore, for the minimum k_1, we obtain

$$(a^2 + c^2 + 2ac)k_1^2 - (1 + b^2 + 2b) = 0 \tag{10.1}$$

or

$$(a^2 + c^2 - 2ac)k_1^2 - (1 + b^2 - 2b) = 0. \tag{10.2}$$

Solving (10.1) and (10.2), we find

$$k_1 = \min[|1 + b|/|a + c|, |1 - b|/|a - c|]$$

From which

$$\sigma_1 = \min\left[\frac{|1 + b}{|a + c|}, \frac{|1 - b|}{|a - c|}\right] - 1. \tag{11}$$

Consequently, interchanging the variables z_1 and z_2, one evaluates

$$\sigma_2 = \min\left[\frac{|1 + a|}{|b + c|}, \frac{|1 - a|}{|b - c|}\right] - 1. \tag{12}$$

The results agree with those of [3]–[8]. Note that here they are derived in a very simple manner. Let us also compute σ. We form the inners matrix of

$$z_1^{N_1} B(kz_1^{-1}, kz_2)$$

This is

$$\Delta_{2N_1}(k, z_2) = \begin{bmatrix} ak + ck^2 z_2 & 1 + bkz_2 \\ 1 + bk\overline{z}_2 & ak + ck^2\overline{z}_2 \end{bmatrix}. \tag{13}$$

Then

$$\det \Delta_{2N_1}(k, z_2) = k^2(a^2+c^2k^2+2ackx)-(1+b^2k^2+2bkx) \tag{14}$$

where $x = \cos\phi_1$. One also obtains that $\det \Delta_{2N_1}(k, z_2)$ is also linear in x. Thus, for a certain k, the minimum value of $\det \Delta_{N_2}(k, z_1)$ is obtained for $x = \pm 1$. Thus, for the minimum k with $\det \Delta_{2N_1}(k, z_2) = 0$ the determinant $\det \Delta_{2N_1}(k, z_2)$ will be zero for $x = \pm 1$. Therefore, we obtain the following for the minimum k:

$$k^2(a^2 + c^2k^2 + 2ack) - (1 + b^2k^2 + 2bk) = 0 \tag{15.1}$$

or

$$k^2(a^2 + c^2k^2 - 2ack) - (1 + b^2k^2 - 2bk) = 0. \tag{15.2}$$

Solving (15.1) and (15.2), we find $k =$ *minimum* of the real positive values of the set

$$\left\{\frac{a+b\pm\sqrt{(a+b)^2-4c}}{2c}, \frac{-a+b\pm\sqrt{(-a+b)^2+4c}}{2c}, \frac{a-b\pm\sqrt{(a-b)^2+4c}}{2c}, \frac{-a-b\pm\sqrt{(-a-b)^2-4c}}{2c}\right\}.$$

From which $\sigma = k - 1$. All the results agree with those derived in [3]–[8], but here they are derived in an easier manner.

Example 2 [6]: Consider $B(z_1, z_2) = 3 - z_1 - z_2$. Following the above procedure, we obtain $\sigma_1 = 1$, $\sigma_2 = 1$, as well as $\sigma = 0.5$. The latter can be obtained from (15.1) and (15.2) if we put $a = b = -1/3$ and $c = 0$.

Remark: An interesting generalization of the definitions of $\sigma_1, \sigma_2, \sigma$ could be the following: *Definition of the stability margin σ with weights*

$$\lambda_1, \lambda_2 \ (\lambda_1 + \lambda_2 = 1, \lambda_1 \geq 0 \ \& \ \lambda_2 \geq 0)$$

Given a 2-D discrete system described by the transfer function (1), we call stability margin σ the supremum of the positive real numbers for which

$$B((1+\lambda_1\sigma)\cdot z_1, (1+\lambda_2\sigma)\cdot z_2)$$

is a (BIBO) Stable Polynomial.

Taking into account this definition, we can consider Definitions 1–3 as special cases of the previous definition (Definition 3 needs a slight modification). Moreover, modifying the above method, one can easily derive a general algorithm for evaluating the stability margin σ with weights λ_1, λ_2.

III. Conclusion

In this brief, the stability margin for 2-D discrete systems has been considered. A new method for computing the stability margins has been proposed. The method is based on a constrained optimization problem of a real positive parameter. Since the formulation of the inners determinant [9] is more "direct" than the formulation of the Schour–Cohn matrix [1], [12], the method, offering a more direct computation of the stability margin, is better than the method of [3].

The significance of the proposed computational method and the improvement with respect to previous work in [3]–[8] is that we use the inners determinant instead of the method of *Schur–Cohn*.

The method of the *inners* determinant has the same multiplexity as the method of the Schur–Cohn ([9]), but it is actually an essential simplification of the Schur–Cohn method as far as the formulation of the various matrices is concerned [9], [12]. For this reason, the proposed method is better than that of [3]–[8].

Work is in progress by the author in the area of 2-D stability-margin formulating analogous methods for 2-D continuous systems. Other recent results can also be found in [2].

Appendix

Consider the mapping

$$\delta\colon\ k_1 \to \delta(k_1) \qquad \text{where} \delta(k_1) = \Delta_{2N_1}(k_1, z_2)$$

This is a *continuous* mapping since the matrix $\Delta_{2N_1}(k_1, z_2)$ consists of polynomials in k_1, z_2. Also, consider the mapping

$$\det\colon\ \delta(k_1) \to \det\delta(k_1)$$

This is also a continuous mapping.

Therefore, their synthesis

$$\det\delta\colon\ k_1 \to \det\delta(k_1)$$

is also a continuous mapping. We denote $\boldsymbol{S}$, the set $\boldsymbol{S} = \{\delta(k_1)$ with $\delta(k_1) > 0\}$, where $>$ denotes positive innerwise for all z_2 with $z_2 = e^{j\phi_2}$ and $\phi_2 \in [0, 2\pi]$ [9]. We also denote $\det\{\boldsymbol{S}\}$ the subset of the real numbers which consists of all the determinants of $\delta(k_1)$ that belong to $\boldsymbol{S}$. Evidently, $\det\{\boldsymbol{S}\}$ is the set of all the (strictly) positive real numbers. Thus, the only limit point of $\det\{\boldsymbol{S}\}$ is the 0.

$\boldsymbol{S}$ is an *open* set and because of the continuity of the mapping δ, the corresponding set of k_1 will also be *open* (see any standard textbook of *Real Analysis* or *Topology* [11]). Thus, the supremum of k_1 is a limit point of this set and because of the continuity of the mapping δ, for this $k_1, \delta(k_1)$ is also a limit point of $\boldsymbol{S}$. Furthermore, by the continuity of the mapping

$$\det\colon\ \delta(k_1) \to \det\delta(k_1), \det\delta(k_1)$$

is the limit point in the set $\det\{\boldsymbol{S}\}$ for this k_1. Since the only limit point of $\det\{\boldsymbol{S}\}$ is the 0, we conclude that for this k_1, we have $\det\ \delta(k_1) = 0$.

As a result, we obtain that for the supremum of k_1 for which $B(k_1z_1, z_2)$ is (*BIBO*) *Stable*, the inners matrix $\Delta_{2N_1}(k_1, z_2)$ will be singular (for some $z_2, z_2 = e^{j\phi_2}$ and $\phi_2 \in [0, 2\pi]$).

References

[1] E. I. Jury, "Stability of multidimensional systems and related problems," in *Multidimensional Systems (Techniques and Applications)*, Spyros G. Tzafestas, Ed. New York: Marcel Dekker, 1986.

[2] N. E. Mastorakis, "Comments concerning multidimensional polynomials properties," *IEEE Trans. Automat. Contr.*, vol. 41, p. 260, Feb. 1996.

[3] P. Agathoklis, E. I. Jury, and M. Mansour, "The margin of stability of 2-D linear discrete systems," *IEEE Trans. Acoust., Speech, Signal Processing*, vol. ASSP-30, pp. 869–873, Dec. 1982.

[4] E. Walach and E. Zeheb, "N-dimensional stability margins computation and a variable transformation," *IEEE Trans. Acoust., Speech, Signal Processing*, vol. ASSP-30, pp. 887–893, Dec. 1982.

[5] W.-S. Lu, A. Antoniou, and P. Agathoklis, "Stability of 2-D digital filters under parameter variations," *IEEE Trans. Circuits Syst.*, vol. CAS-33, pp. 476–482, May 1986.

[6] L. M. Roytman, M. N. S. Swamy, and G. Eichmann, "An efficient numerical scheme to compute 2-D stability thresholds," *IEEE Trans. Circuits Syst.*, vol. CAS-34, pp. 322–324, Mar. 1987.

[7] M. Wang, E. B. Lee, and D. Booley, "A simple method to determine the stability and the margin of stability of 2-D recursive filters," *IEEE Trans. Circuits Syst. I*, vol. 39, pp. 237–239, Mar. 1992.

[8] D. Hertz and E. Zeheb, "Simplification in multidimensional stability margin computations," *IEEE Trans. Acoust., Speech, Signal Processing*, vol. ASSP-35, pp. 566–568, Apr. 1987.

[9] A. J. Kanellakis, S. G. Tzafestas, and N. J. Theodorou, "Stability tests for 2-D systems using the Schwarz form and inners determinants," *IEEE Trans. Circuits Syst.*, vol. 38, pp. 1071–1077, Sept. 1991.

[10] R. P. Roesser, "A discrete state-space model for image processing," *IEEE Trans. Automat. Contr.*, vol. AC-20, pp. 1–10, Feb. 1975.

[11] W. Sutherland, *Introduction to Metric and Topological Spaces*. Oxford, U.K.: Clarendon, 1975.

[12] E. I. Jury, "Modified stability table for 2-D digital filters," *IEEE Trans. Circuits Syst.*, vol. 35, pp. 116–119, Jan. 1988.

A Global Least Mean Square Algorithm for Adaptive IIR Filtering

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***Abstract*— In this brief, we develop an least mean square (LMS) algorithm that converge in a statistical sense to the global minimum of the mean square error (MSE) objective function. This is accomplished by estimating the gradient as a smoothed version of the MSE. The smoothed MSE objective begins as a convex functional in the mean. The amount of dispersion or smoothing is reduced, such that over time it becomes the true MSE as the algorithm converges to the global minimum. We show that this smoothing behavior is approximated by appending a variable noise source to the infinite impulse response (IIR)–LMS algorithm. We show, experimentally, that the proposed method does converge to the global minimum in the cases tested. A performance improvement over the IIR–LMS algorithm and the Steiglitz–McBride algorithm has been achieved.**

I. Introduction

Adaptive filtering represents a major research area in digital signal processing, communications, and control. There exist many applications of adaptive filtering in communications and signal processing that require filters that self-modify, based on the signals encountered within their operating environment. Examples of important applications include linear prediction, adaptive differential pulse coding, echo cancellation, channel equalization, and system identification [6].

Adaptive filters based upon the finite-impulse response (FIR) structure have matured to a point of practical implementations. A major drawback of the adaptive FIR filter is that certain applications will require a very large number of parameters to achieve good performance, thus, increasing computational costs. This becomes evident when the system to be modeled or identified is represented as a pole–zero model.

On the other hand, adaptive filters based upon the infinite-impulse response (IIR) structure [4] have the advantage of approximating a pole–zero model more accurately than the FIR structure. This increased accuracy can be accomplished with an equivalent-order IIR filter, thereby reducing the computational cost in terms of the number of coefficients to be estimated. Although adaptive IIR filters require less coefficients to be estimated, the system may become unstable during adaptation. Another problem area is that the objective function for an adaptive IIR filter can be nonconvex, which implies the existence of multiple local minima. Adaptive IIR filtering typically uses gradient search techniques, e.g., the least mean square (LMS) algorithms [6], which are sensitive to initial conditions. Therefore, these techniques can easily converge to a local minimum, resulting in a suboptimal solution. Hence, adaptive IIR filters are not used commercially at this time. Additional open problems of adaptive IIR filtering, such as convergence to an unstable filter, are discussed in [1] and [2].

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We propose to address the problem of convergence to a local minimum of an adaptive IIR filter by investigating the use of stochastic global-optimization methods. This type of global optimization procedure has the property of converging to the global minimum with a probability of one, as the number of iterations tends to infinity [21]. One such general method is the stochastic approximation method, which represents a simple approach to minimizing a nonconvex function. This method is based on using a randomly distributed process to find the absolute minimum of an objective function [3], [10], [16]. In particular, stochastic approximation with convolution smoothing (SAS) has been successfully used as a global optimization algorithm in several applications [5], [11], [12]. Though similar to simulated annealing [13], SAS was empirically proven to be more efficient computationally and more accurate in converging to a global minimum [5]. The objective of convolution smoothing is to "smooth" the nonconvex objective function by convolving it with a noise probability density function (pdf). The variance on the pdf at the start of the optimization procedure is large, which has the effect of "smoothing" the objective function so that it is convex. Then the variance is slowly reduced to zero, whereby the smooth functional returns to the original objective function, as the algorithm converges to the global minimum.

The SAS method represents an off-line procedure for optimizing deterministic objective functions where the data is static and, therefore, is not conducive for adaptive filtering. We will develop an on-line approximation of this method for time-series data. The proposed method is developed from the SAS algorithm by first showing that an on-line version of the algorithm computes the gradient at the present location perturbed by a random value. Secondly, we approximate this gradient by an instantaneous function of its Taylor series expansion. Combining this approximation of the gradient with the LMS algorithm results in a stochastic global optimization algorithm for adaptive IIR filtering. The resultant global optimization LMS algorithm consists of the standard LMS algorithm with the addition of a noise term, whose variance is initially large and approaches zero as the iteration progresses in time. This formulation only incrementally increases the computational cost of the LMS algorithm. Experimentally, we show that the proposed algorithm converges to the global minimum, thereby, alleviating a major problem of adaptive IIR filtering.

This brief is organized as follows: Section II is an overview of the general SAS algorithm, Section III develops the global least-mean-square (GLMS) algorithm for adaptive IIR filtering. Shown in Section IV are the experimental results of using the GLMS method for identifying an unknown system, along with a comparison of its behavior to the IIR–LMS algorithm [6] and the Steiglitz–McBride algorithm [17], and Section V gives concluding remarks.

II. Background

SAS is an unconstrained global-optimization algorithm for minimizing a nonconvex function

$$\min_{x \in R^n} g(x). \tag{1}$$